# Chapter 2

2.1 Let $x \in (X \setminus C) \cap D$. Then $x \in X, x \in D, x \notin C$. So $x \in D, x \notin C$ which gives $x \in D \setminus C$. Hence $(X \setminus C) \cap D \subseteq D \setminus C$.

Conversely, if $x \in D \setminus C$ then $x \notin C$ so $x \in X \setminus C$ , and also $x \in D$. So $x \in (X \setminus C) \cap D$. Hence $D \setminus C \subseteq (X \setminus C) \cap D$.

Together these prove that $(X \setminus C) \cap D = D \setminus C$.

2.3 Suppose that $x \in V$. Then $x \in X$ and $x \notin X \setminus V = X \cap U$, so $x \notin U$. So $x \in X \subseteq Y$ and $x \notin U$, so $x \in Y \setminus U$. This gives $x \in X \cap (Y \setminus U)$. Hence $V \subseteq X \cap (Y \setminus U)$.

Conversely suppose that $x \in X \cap (Y \setminus U)$. Then $x \in X$ and $x \notin U$, so $x \notin X \cap U = X \setminus V$. Hence $x \in V$. This shows that $X \cap (Y \setminus U) \subseteq V$.

Together these show that $V = X \cap (Y \setminus U)$.

2.5 If $(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$ then $x \in U_1$ and $x \in U_2$ so $x \in U_1 \cap U_2$, and similarly $y \in V_1 \cap V_2$, so $(x, y) \in (U_1 \cap U_2) \times (V_1 \cap V_2)$. This shows that

$$(U_1 \times V_1) \cap (U_2 \times V_2) \subseteq (U_1 \cap U_2) \times (V_1 \cap V_2).$$

Conversely, if $(x, y) \in (U_1 \cap U_2) \times (V_1 \cap V_2)$ then $x \in U_1, x \in U_2$ and $y \in V_1, y \in V_2$ so $(x, y) \in U_1 \times V_1$ and also $(x, y) \in U_2 \times V_2$, so $(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$. This shows that

$$(U_1 \cap U_2) \times (V_1 \cap V_2) \subseteq (U_1 \times V_1) \cap (U_2 \times V_2).$$

Together these show that

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2).$$

2.7 (a) Let the distinct equivalence classes be $\{A_i : i \in I\}$. Each $A_i$, being an equivalence class, satisfies $A_i \subseteq X$. To see that the distinct equivalence classes are disjoint, suppose that for some $i, j \in I$ and some $x \in X$ we have $x \in A_i \cap A_j$. Then for any $a \in A_i$ we have $a \sim x$ and $x \in A_j$, hence $a \in A_j$. This shows that $A_i \subseteq A_j$. Similarly $A_j \subseteq A_i$. This shows that $A_i = A_j$. Thus distinct equivalence classes are mutually disjoint. Finally, any $x \in X$ is in some equivalence class with respect to $\sim$, so $X \subseteq \bigcup_{i \in I} A_i$. Also, since each $A_i$ is a subset of $X$ we have $\bigcup_{i \in I} A_i \subseteq X$.

So $X = \bigcup_{i \in I} A_i$ and the union on the right-hand side is disjoint.

(b) We define $x_1 \sim x_2$ iff $x_1, x_2 \in A_i$ for some $i \in I$. This is reflexive since each $x \in X$ is in some $A_i$ so $x \sim x$. It is symmetric since if $x_1 \sim x_2$ then $x_1, x_2 \in A_i$ for some $i \in I$ and then also $x_2, x_1 \in A_i$ so $x_2 \sim x_1$. Finally it is transitive since if $x_1 \sim x_2$ and $x_2 \sim x_3$ then $x_1, x_2 \in A_i$ for some $i \in I$ and $x_2, x_3 \in A_j$ for some $j \in I$. Now $x_2 \in A_i \cap A_j$ and since $A_i \cap A_j = \emptyset$ for $i \neq j$ we must have $j = i$. Hence $x_1, x_3 \in A_i$ and we have $x_1 \sim x_3$ as required for transitivity.

# Chapter 3

3.1 Suppose that $y \in f(A)$. Then $y = f(a)$ for some $a \in A$. Since $A \subseteq B$, also $a \in B$ so $y = f(a) \in f(B)$. By definition $f(B) \subseteq Y$. This shows that $f(A) \subseteq f(B) \subseteq Y$.

Suppose that $x \in f^{-1}(C)$. Then $f(x) \in C$, so since $C \subseteq D$ we have also that $f(x) \in D$. Hence $x \in f^{-1}(D)$. By definition $f^{-1}(D) \subseteq X$. This shows that $f^{-1}(C) \subseteq f^{-1}(D) \subseteq X$.

3.3 First suppose that $x \in (g \circ f)^{-1}(U)$, so $g(f(x)) = (g \circ f)(x) \in U$. Hence by definition of inverse images $f(x) \in g^{-1}(U)$ and again by definition of inverse images $x \in f^{-1}(g^{-1}(U))$. This shows that $(g \circ f)^{-1}(U) \subseteq f^{-1}(g^{-1}(U))$.

Now suppose that $x \in f^{-1}(g^{-1}(U))$. Then $f(x) \in g^{-1}(U)$, so $g(f(x)) \in U$, that is $(g \circ f)(x) \in U$, and by definition of inverse images, $x \in (g \circ f)^{-1}(U)$. Hence $f^{-1}(g^{-1}(U)) \subseteq (g \circ f)^{-1}(U)$.

These together show that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$.

3.5 We know from Proposition 3.14 in the book that if $f : X \to Y$ is onto then $f(f^{-1}(C)) = C$ for any subset $C$ of $Y$.

Suppose that $f : X \to Y$ is such that $f(f^{-1}(C)) = C$ for any subset $C$ of $Y$. For any $y \in Y$ we can put $C = \{y\}$ and get that $f(f^{-1}(y)) = \{y\}$. This tells us that there exists $x \in f^{-1}(y)$ (for which of course $f(x) = y$ ), so $f^{-1}(y) \neq \emptyset$. This proves that $f$ is onto.

3.7 (i) We can have $y \neq y'$ with neither $y$ nor $y'$ in the image of $f$ , so that $f^{-1}(y) = f^{-1}(y') = \emptyset$. For a counterexample, we may define $f : \{0\} \to \{0, 1, 2\}$ by $f(0) = 0$ and take $y = 1, y' = 2$.

(ii) Suppose that $f : X \to Y$ is onto and $y, y' \in Y$ with $y \neq y'$, and suppose for a contradiction that $f^{-1}(y) = f^{-1}(y')$. Since $f$ is onto, there exists $x \in f^{-1}(y) = f^{-1}(y')$. This gives $y = f(x) = y'$, a contradiction. Hence (ii) is true.

3.9 (a) Suppose that $y \in f(A) \cap C$. Then $y \in C$ and $y = f(a)$ for some $a \in A$. Then $a \in f^{-1}(C)$, so $a \in A \cap f^{-1}(C)$ and $y = f(a) \in f(A \cap f^{-1}(C))$. Hence $f(A) \cap C \subseteq f(A \cap f^{-1}(C))$.

Conversely suppose $y \in f(A \cap f^{-1}(C))$. Then $y = f(a)$ for some $a \in A \cap f^{-1}(C)$. Then $y \in f(A)$ since $a \in A$ and $y = f(a) \in C$ since $a \in f^{-1}(C)$. Hence $f(A \cap f^{-1}(C)) \subseteq f(A) \cap C$.

Together these show that $f(A) \cap C = f(A \cap f^{-1}(C))$.

(b) We apply (a) with $C = f(B)$. This shows that

$f(A) \cap f(B) = f(A \cap f^{-1}(f(B))$, so since $f^{-1}(f(B)) = B$ we have $f(A) \cap f(B) = f(A \cap B)$.

# Chapter 4

4.1 Suppose that $u$ is an upper bound for $B$. Then since $A \subseteq B$ we have $a \leqslant u$ for all $a \in A$. So $A$ is bounded above. In particular since $\sup B$ is *an* upper bound for $B$ it is also an upper bound for $A$. Hence $\sup A \leqslant \sup B$.

4.3 (a) We prove that if $\emptyset \neq A \subseteq B$ and if $B$ is bounded below then $A$ is bounded below and $\inf A \geqslant \inf B$. For if $l$ is a lower bound for $B$ then $a \geqslant l$ for all $a \in A$ since $A \subseteq B$. So $A$ is bounded below. In particular $\inf B$ is *a* lower bound for $A$, so $\inf A \geqslant \inf B$.

(b) We prove that if $A$ and $B$ are non-empty subsets of $\mathbb{R}$ which are bounded below, then $A \cup B$ is bounded below and $\inf(A \cup B) = \min\{\inf A, \inf B\}$. For let $l = \min\{\inf A, \inf B\}$. If $x \in A \cup B$ then either $x \in A$ so $x \geqslant \inf A \geqslant l$, or $x \in B$ so $x \geqslant \inf B \geqslant l$. In either case $x \geqslant l$. Hence $l$ is a lower bound for $A \cup B$. So $A \cup B$ is bounded below and $\inf(A \cup B) \geqslant l$. Now let $\varepsilon > 0$. If $l = \inf A$ then there exists $x \in A$ such that $x < l + \varepsilon$, and if $f = \inf B$ then there exists $x \in B$ such that $x < l + \varepsilon$. In either case there exists $x \in A \cup B$ such that $x < l + \varepsilon$. Hence $l$ is the greatest lower bound of $A \cup B$. We now have $\inf(A \cup B) = \min\{\inf A, \inf B\}$ as required.

4.5 Suppose for a contradiction that $q^2 = 2$ where $q = m/n$, with $m, n$ mutually prime integers. Then $m^2 = 2n^2$. Now $2$ divides the right-hand side of this equation, hence $2$ divides $m^2$ (we write $2|m^2$ ). Since $2$ is prime, we must have $2|m$. So in fact $4|m^2$, and from the equation $m^2 = 2n^2$ again we get $2|n^2$ so $2|n$. But now $2|m$ and $2|n$ together contradict the hypothesis that $m, n$ are mutually prime. Hence there is no such rational number $q$.

4.7 Suppose that $S$ is a non-empty set of real numbers which is bounded below, say $s \geqslant k$ for all $s \in S$. Let $-S$ mean the set $\{x \in \mathbb{R} : -x \in S\}$. Then for any $x \in -S$ we have $-x \geqslant k$ so $x \leqslant -k$. This shows that $-S$ is bounded above, so by the completeness property $-S$ has a least upper bound $\sup(-S)$. Put $l = -\sup(-S)$. For any $y \in S$ we have $-y \in -S$ so $-y \leqslant \sup(-S)$, whence $y \geqslant -\sup(-S) = l$. Thus $l$ is a lower bound for $S$.

Now let $l'$ be any lower bound for $S$ , so that $y \geqslant l'$ for any $y \in S$. Then $-y \leqslant -l'$ for any $y \in S$, which says that $x \leqslant -l'$ for any $x \in -S$. Thus $-l'$ is an upper bound for $-S$ , and by leastness of $\sup(-S)$ we have $-l' \geqslant \sup(-S)$. This gives $l' \leqslant -\sup(-S) = l$. So $l$ is a greatest lower bound for $S$.

4.9 Since $y > 1$ we have $y = 1 + x$ for some $x > 0$. Hence $y^n = (1+x)^n$. Choose some integer $r$ with $r > \alpha$ and let $n > r$. Then

$$(1+x)^n > \frac{n(n-1)(n-2)\dots(n-r+1)x^r}{r!},$$

so $$0 \leqslant \frac{n^\alpha}{y^n} < \frac{r!n^\alpha}{n(n-1)(n-2)\dots(n-r+1)x^r} \to 0$$

as $n \to \infty$, since there are $r$ factors on the denominator involving $n$, and $r > \alpha$. The result now follows by the 'sandwich principle'.

4.11 Suppose $a = a_{i_0}$. Then $a^n = a_{i_0}^n \leqslant a_1^n + a_2^n + \ldots + a_r^n$. Also, $a_i \leqslant a$ for any $i \in \{1, 2, \ldots, r\}$ so $a_i^n \leqslant a^n$. Hence $a_1^n + a_2^n + \ldots + a_r^n \leqslant ra^n$. As the hint suggests we now take $n$ th roots and get

$$a \leqslant (a_1^n + a_2^n + \ldots + a_r^n)^{1/n} \leqslant r^{1/n} a.$$

Now $r^{1/n} \to 1$ as $n \to \infty$ (this follows from Exercise 4.10, since $1 < r^{1/n} < n^{1/n}$ for all $n > r$ ). So by the sandwich principle for limits $(a_1^n + a_2^n + \ldots + a_r^n)^{1/n} \to a$ as $n \to \infty$.

4.13 (a) If $y \geqslant z$ then $\max\{y, z\} = y$ and $|y-z| = y-z$ so $(y+z+|y-z|)/2 = y$. If $y < z$ then $\max\{y, z\} = z$ and $|y-z| = z-y$ so $(y+z+|y-z|)/2 = z$.

If $y \geqslant z$ then $\min\{y, z\} = z$ and $|y-z| = y-z$ so $(y+z-|y-z|)/2 = z$. If $y < z$ then $\min\{y, z\} = y$ and $|y-z| = z-y$ so $(y+z-|y-z|)/2 = y$.

These prove (a).

(b) We use (a) to see that for each $x \in \mathbb{R}$ we have

$$h(x) = \frac{1}{2}(f(x) + g(x) + |f(x) - g(x)|), \;\; k(x) = \frac{1}{2}(f(x) + g(x) - |f(x) - g(x)|).$$

Now $f$ and $g$ are continuous, hence $f+g$ and $f-g$ are continuous by Proposition 4.31 (we note that the constant function with value $-1$ is continuous, hence $-g$ is continuous since $g$ is continuous). Hence, again by Proposition 4.31, $|f-g|$ is continuous. Hence $f+g \pm |f-g|$ is continuous, so $h, k$ are continuous.

4.15 Let $x \in \mathbb{R}$ and take $\varepsilon = 1/2$. If $f$ were continuous at $x$ there would exist $\delta > 0$ such that $|f(x) - f(y)| < 1/2$ for any $y \in \mathbb{R}$ such that $|y - x| < \delta$. Now we know from Corollary 4.7 and Exercise 4.8 that there exist both a rational number $x_1$ and an irrational number $x_2$ between $x$ and $x+\delta$. Thus $|x - x_1| < \delta$ and $|x - x_2| < \delta$. Hence we should have $|f(x) - f(x_1)| < 1/2$ and $|f(x) - f(x_2)| < 1/2$ , so

$$|f(x_1) - f(x_2)| \leqslant |f(x_1) - f(x)| + |f(x) - f(x_2)| < 1.$$

But in fact $f(x_1) = 0$ and $f(x_2) = 1$, so $|f(x_1) - f(x_2)| = 1$. This contradiction shows that $f$ is not continuous at $x$.

4.17 We use the fact that the graph of a convex function is convex, that is if $x, y$ are real numbers with $x < y$ then the straight-line segment joining the points $(x, f(x))$ and $(y, f(y))$ lies above or on the graph of $f$ between $x$ and $y$. (See Figure 1 below.)

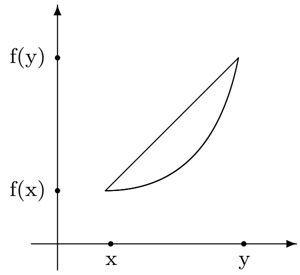



Figure 1: Convexity

Any point on this straight-line segment is of the form

$$(\lambda x + (1-\lambda)y, \, \lambda f(x) + (1-\lambda)f(y)) \text{ for some } \lambda \in [0, 1].$$

Now the graph of $f$ at the point $\lambda x+(1-\lambda)y$ has height $f(\lambda x+(1-\lambda)y)$ and the definition of convexity says that this height is not greater than $\lambda f(x)+(1-\lambda)f(y)$. For any point $a \in \mathbb{R}$ choose $b < a$ and $c > a$, so $a = \lambda b + (1-\lambda)c$ for some $\lambda \in (0, 1)$. Let $L_1$ be the straight line through the points $(a, f(a))$ and $(c, f(c))$ and let $L_2$ be the straight line through $(b, f(b))$ and $(a, f(a))$ (see Figure 2). The idea of the proof is that by convexity the graph of $f$ on $[b, c]$ is trapped in the double cone formed by the lines $L_1, L_2$ and from this we can deduce continuity of $f$ at $a$.

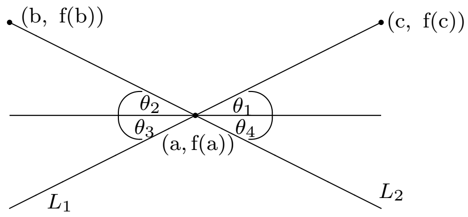



Figure 2: Convex continuity

First, by convexity applied on $[a, c]$ for any $x \in (a, c)$ the point $(x, f(x))$ is below or on $L_1$. Less obvious but also true is the fact that $(x, f(x))$ is above or on $L_2$. This follows from convexity applied between $b$ and $x$ : if $(x, f(x))$ were below $L_2$ then $(a, f(a))$ would be above the straight-line segment joining $(b, f(b))$ to $(x, f(x))$, contradicting convexity. By a similar argument we can show that if $x \in (b, a)$ then the point $(x, f(x))$ lies below $L_2$ and above $L_1$.

Now to prove continuity at $a$, , let $\theta_1, \theta_2, \theta_3, \theta_4$ be the angles indicated in Figure 2. Given $\varepsilon > 0$ choose a positive $\delta < \varepsilon/M$ where

$$M = \max\{|\tan\theta_1|, |\tan\theta_2|, |\tan\theta_3|, |\tan\theta_4|\}.$$

Then for any $x$ satisfying $|x-a| < \delta$ we have $|f(x)-f(a)| \leqslant M\delta$ and so $|f(x)-f(a)| < \varepsilon$ as required.

# Chapter 5

5.1 From the triangle inequality $d(x, z) \leqslant d(x, y)+d(y, z)$ so $d(x, z) - d(y, z) \leqslant d(x, y)$. From the triangle inequality and symmetry $d(y, z) \leqslant d(y, x) + d(x, z) = d(x, y) + d(x, z)$, so $(y, z) - d(x, z) \leqslant d(x, y)$. Together these give $|d(x, z) - d(y, z)| \leqslant d(x, y)$.

5.3 The proof is by induction on $n$. For $n = 3$ it is the triangle inequality. Suppose it is true for a given integer value of $n \geqslant 3$. Then, using the triangle inequality and the inductive hypothesis we get that $d(x_1, x_{n+1})$ is less than or equal to

$$d(x_1, x_n) + d(x_n, x_{n+1}) \leqslant d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n) + d(x_n, x_{n+1}),$$

which tells us the formula is true for $n+1$. By induction it is true for all integers $n \geqslant 3$.

5.5 Suppose for a contradiction that $z \in B_\varepsilon(x) \cap B_\varepsilon(y)$. Then $d(z, x) < \varepsilon$ and $d(z, y) < \varepsilon$, so by the triangle inequality and symmetry we get

$$d(x, y) \leqslant d(x, z) + d(z, y) = d(z, x) + d(z, y) < 2\varepsilon.$$

This contradicts the fact that $d(x, y) = 2\varepsilon$.

5.7 Since $S$ is bounded, we have for some $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and some $K \in \mathbb{R}$

$\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + \ldots + (x_n - a_n)^2} \leqslant K$ for all $(x_1, x_2, \ldots, x_n) \in S$.

In particular, for each $i = 1, 2, \ldots, n$, $|x_i - a_i| \leqslant K$ so $x_i \in [a_i - K, a_i + K]$. Let $a = m - K$ and $b = M + K$ where we define $m = \min\{a_i : i = 1, 2, \ldots, n\}$, and $M = \max\{a_i : i = 1, 2, \ldots, n\}$. Then $x_i \in [a, b]$ for each $i = 1, 2, \ldots, n$ so $(x_1, x_2, \ldots, x_n) \in [a, b] \times [a, b] \times \ldots \times [a, b]$ (product of $n$ copies of $[a, b]$ ). This holds for all $(x_1, x_2, \ldots, x_n)$ in $S$, so $S \subseteq [a, b] \times [a, b] \times \ldots \times [a, b]$ (product of $n$ copies of $[a, b]$ ).

5.9 Let the metric space be $X$. There exists some $x_0 \in X$ and some $K \in \mathbb{R}$ such that $d(b, x_0) \leqslant K$ for all $b \in B$. Since $A \subseteq B$ this holds in particular for all points in $A$ , so $A$ is bounded.

If $A = \emptyset$ then by definition diam $A = 0$ so diam $A \leqslant$ diam $B$, since the latter is the sup of a set of non-negative real numbers. Now suppose $A \neq \emptyset$. Since $d(b, b') \leqslant$ diam $B$ for all $b, b' \in B$, in particular $d(a, a') \leqslant$ diam $B$ for any $a, a' \in A$. Since diam $A$ is the sup of such distances, diam $A \leqslant$ diam $B$.

5.11 Since $d_\infty((x, y), (0, 0)) = \max\{|x|, |y|\}$, $(x, y) \in B_1^{d_\infty}((0, 0))$ iff $-1 < x < 1$ and also $-1 < y < 1$, so the unit ball is the interior of the square shown in Figure 3 below.

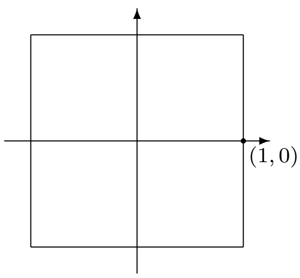



Figure 3

5.13 Since any open ball is an open set by Proposition 5.31, any union of open balls is an open set by Proposition 5.41.

Conversely, given an open set in a metric space $X$, for each $x \in U$ there exists by definition $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \subseteq U$. Then $U = \bigcup_{x \in U} B_{\varepsilon_x}(x)$. For since each $B_{\varepsilon_x}(x) \subseteq U$, their union is contained in $U$. Also, any $x \in U$ is in $B_{\varepsilon_x}(x)$ and hence is in the union on the right-hand side.

5.15 (a) If $y \in B^{d'}_{\varepsilon/k}(x)$ then we have $d'(y, x) < \varepsilon/k$, so $d(x, y) \leqslant kd'(x, y) < \varepsilon$ and $y \in B^d_\varepsilon(x)$, showing that $B^{d'}_{\varepsilon/k}(x) \subseteq B^d_\varepsilon(x)$.

(b) If $U$ is $d$-open then for any $x \in U$ there exists $\varepsilon > 0$ such that $B^d_\varepsilon(x) \subseteq U$. Then $B^{d'}_{\varepsilon/k}(x) \subseteq B^d_\varepsilon(x) \subseteq U$. So $U$ is $d'$-open.

(c) This follows from (b) together with Exercise 5.14.

5.17 Let $x,\, y,\, z,\, t \in X$. From Exercise 5.2,

$$|d(x,\, y) - d(z,\, t)| \leqslant d(x,\, z) + d(y,\, t) = d_1((x,\, y),\, (z,\, t)).$$

So given $\varepsilon > 0$ we may take $\delta = \varepsilon$ and if $d_1((x,\, y),\, (z,\, t)) < \delta$ then

$$|d(x,\, y) - d(z,\, t)| < \delta = \varepsilon, \text{ so } d : X \times X \to \mathbb{R} \text{ is continuous.}$$

# Chapter 6

6.1 (a) (i) The complement of $[a,\, b]$ in $\mathbb{R}$ is $(-\infty,\, a) \cup (b,\, \infty)$, a union of open intervals which is therefore open in $\mathbb{R}$. So $[a,\, b]$ is closed in $\mathbb{R}$.

(ii) The complement of $(-\infty,\, 0]$ in $\mathbb{R}$ is $(0,\, \infty)$ which is open in $\mathbb{R}$ so $(-\infty,\, 0]$ is closed in $\mathbb{R}$.

(iii) The complement $(-\infty,\, 0) \cup (0, \infty)$ is open in $\mathbb{R}$ so $\{0\}$ is closed in $\mathbb{R}$.

(iv) The complement is $(-\infty,\, 0) \cup (1,\, \infty) \cup \bigcup_{n \in \mathbb{N}} (1/(n+1),\, 1/n)$ which is open in $\mathbb{R}$ so this set is closed in $\mathbb{R}$.

(b) The complement of the closed unit disc in $\mathbb{R}^2$ is $S = \{(x_1,\, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 > 1\}$. If $(x_1,\, x_2) \in S$, let us put $\varepsilon = \sqrt{x_1^2 + x_2^2} - 1$. Then $B_\varepsilon((x_1,\, x_2)) \subseteq S$ since if $(y_1,\, y_2)$ is in $B_\varepsilon((x_1,\, x_2))$ then writing $0,\, x,\, y$ for $(0,\, 0),\, (x_1,\, x_2),\, (y_1,\, y_2)$ respectively, we have from the reverse triangle inequality

$$d(0,\, y) \geqslant d(0,\, x) - d(x\,, y) > d(0,\, x) - \varepsilon = \sqrt{x_1^2 + x_2^2} - \varepsilon = 1,$$

so $y \in S$. Hence $S$ is open in $\mathbb{R}^2$ so the closed unit disc is closed in $\mathbb{R}^2$.

(c) Let $S$ be the complement of this rectangle $R$ in $\mathbb{R}^2$. Then $S$ may be written as the union of the four sets

$$U_1 = \mathbb{R} \times (-\infty,\, c),\;\; U_2 = \mathbb{R} \times (d,\, \infty),\;\; U_3 = (-\infty,\, a) \times \mathbb{R},\;\; U_4 = (b,\, \infty) \times \mathbb{R}.$$

Each of these is open in $\mathbb{R}^2$. For example if $(x_1,\, x_2) \in U_1$ then $x_2 < c$. Take $\varepsilon = c - x_2$. We shall prove that $B_\varepsilon((x_1,\, x_2)) \subseteq U_1$. For if $(y_1,\, y_2) \in B_\varepsilon((x_1,\, x_2))$ then $|y_2 - x_2| < \varepsilon = c - x_2$ so $y_2 - x_2 < c - c_2$ which gives $y_2 < c$ so $(y_1,\, y_2) \in U_1$ as claimed. This shows that $U_1$ is open in $\mathbb{R}^2$. Similar arguments show that $U_2,\, U_3,\, U_4$ are open in $\mathbb{R}^2$. Hence $S = U_1 \cup U_2 \cup U_3 \cup U_4$ is open in $\mathbb{R}^2$, and $R$ is therefore closed in $\mathbb{R}^2$.

(d) In a discrete metric space $X$ any subset of $X$ is open in $X$ ; in particular the complement in $X$ of any set $C$ is open in $X$ so $C$ is closed in $X$.

(e) It was proved on p.61 of the book that this subset is closed in $\mathcal{C}([0,\, 1])$.

6.3 The complement of a singleton set $\{x\}$ in a metric space $(X,\, d)$ is open in $X$, for if $y \neq x$ then $B_\varepsilon(y) \subseteq X \setminus \{x\}$ where $\varepsilon = d(x,\, y)$. So $\{x\}$ is closed in $X$. The union of a finite number of sets closed in $X$ is also closed in $X$ by Proposition 6.3, and the result follows.

6.5 Since $C_n$ is the union of a finite number (namely $2^n$ ) of closed intervals, $C_n$ is closed in $\mathbb{R}$ for each $n \in \mathbb{N}$ so $C$ is closed in $\mathbb{R}$ by Exercise 6.4.

6.7 We note first that $\{0, 1\}$ are points of closure of each of these intervals, since given any $\varepsilon > 0$ there is a point of each of these intervals in $B_\varepsilon(i)$ for $i = 1, 2$.

But also, if $x \notin [0, 1]$ then either $x < 0$ or $x > 1$, and in either case there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \cap [0, 1] = \emptyset$, so $x$ is not a point of closure of $[0, 1]$. This completes the proof.

6.9 Suppose that $A$ is a non-empty subset of $\mathbb{R}$ which is bounded above and let $u = \sup A$. Take any $\varepsilon > 0$. Then by leastness of $u$ there is some $a \in A$ with $a > u - \varepsilon$. Since $u$ is an upper bound for $A$, also $a \leqslant u$. So $a \in A \cap B_\varepsilon(u)$. this shows that $u \in \overline{A}$. The proof for inf is similar.

6.11 The sets in Exercise 6.2 (a) and (d) are closed in $\mathbb{R}$ so by Proposition 6.11 (c) they are their own closures in $\mathbb{R}$. The closure of the set in (b) is $\mathbb{R}$ : for given any $x \in \mathbb{R}$ and any $\varepsilon > 0$ there exists an irrational number in (for example) $(x - \varepsilon, x)$ by Exercise 4.8. The closure of the set $A$ in 6.11(c) is $A \cup \{1\}$ : for given any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ with $1/(n+1) < \varepsilon$, which says that $1 - n/(n+1) < \varepsilon$, and this shows that $1 \in \overline{A}$. Also, no point in the complement of $A \cup \{1\}$ is in $\overline{A}$ : for the complement of $A \cup \{1\}$ in $\mathbb{R}$ is

$$\left(-\infty, \frac{1}{2}\right) \cup (1, \infty) \cup \bigcup_{n=1}^{\infty} \left(\frac{n}{n+1}, \frac{n+1}{n+2}\right), \text{ which is open in } \mathbb{R}.$$

The closure of the set in (d) is the set itself, since it is closed - its only limit point in $\mathbb{R}$ is 0.

6.13 First suppose that $f : X \to Y$ is continuous. Let $y \in f(\overline{A})$ for some $A \subseteq X$ and let $\varepsilon > 0$. Then $y = f(x)$ for at least one $x \in \overline{A}$. By continuity of $f$ at $x$ there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(y)$. By definition of $\overline{A}$ there exists some $a \in A \cap B_\delta(x)$. Then $f(a) \in f(B_\delta(x)) \subseteq B_\varepsilon(y)$. So $f(a)$ is in $B_\varepsilon(y) \cap f(A)$ which shows that $y \in \overline{f(A)}$. Hence $f(\overline{A}) \subseteq \overline{f(A)}$.

Conversely suppose that $f(\overline{A}) \subseteq \overline{f(A)}$ for any subset $A$ of $X$. We shall prove that the inverse image of any closed subset $V$ of $Y$ is closed in $X$ , so that $f$ is continuous by Proposition 6.6. For suppose that $V$ is closed in $Y$. We have

$$f(\overline{f^{-1}(V)}) \subseteq \overline{f(f^{-1}(V)}, \text{ and } f(f^{-1}(V) \subseteq V \text{ so } \overline{f(f^{-1}(V)} \subseteq \overline{V} = V,$$

where the last equality follows from Proposition 6.11 (c) since $V$ is closed in $Y$. Hence $f(\overline{f^{-1}(V)} \subseteq V$, so $\overline{f^{-1}(V)} \subseteq f^{-1}(V)$. Since we always have the other inclusion $f^{-1}(V) \subseteq \overline{f^{-1}(V)}$, this shows that $\overline{f^{-1}(V)}$ equals $f^{-1}(V)$, and $f^{-1}(V)$ is closed in $X$ by Proposition 6.11 (c).

6.15 Since for each $i \in I$ we have that $A_i \subseteq \overline{A_i}$, it follows that $\bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} \overline{A_i}$.

But each $\overline{A_i}$ is closed in $X$ by Proposition 6.11 (c) so $\bigcap_{i \in I} \overline{A_i}$ is closed in $X$ by Proposition 6.4, and hence by Proposition 6.11 (f) $\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}$.

In $\mathbb{R}$ we may take $A_1 = (0,\, 1)$, $A_2 = (1,\, 2)$. Then $A_1 \cap A_2 = \emptyset$ so $\overline{A_1 \cap A_2} = \emptyset$, whereas $\overline{A_1} = [0,\, 1]$ and $\overline{A_2} = [1,\, 2]$ so $\overline{A_1} \cap \overline{A_2} = \{1\}$.

6.17 (a) Each point in $A = [1,\, \infty)$ is a limit point of $A$ , since given any $x \in \mathbb{R}$ with $1 \leqslant x$ and any $\varepsilon > 0$, the open ball $(x - \varepsilon,\, x + \varepsilon)$ contains points of $A$ other than $x$ (for example $x + \varepsilon/2$ ). Also, no point in the complement of $A$ is a limit point of $A$ , since any limit point of $A$ is in $\overline{A}$, and we have seen in Exercise 6.9 that $\overline{A} = A$. So the set of limit points of $A$ in $\mathbb{R}$ is precisely $A$.

(b) Any real number is a limit point of $\mathbb{R} \backslash \mathbb{Q}$ in $\mathbb{R}$ , since given $x \in \mathbb{R}$ and $\varepsilon > 0$, by Exercise 4.8 there is an irrational number for example in $(x - \varepsilon,\, x)$ and this is not equal to $x$. So the set of limit points here is $\mathbb{R}$.

(c) We know from Definition 6.15 that any limit point of $A$ in $\mathbb{R}$ is a point of closure of $A$, and we have seen in Exercise 6.9 that $\overline{A} = A \cup \{1\}$. Now $1$ is a limit point of $A$ in $\mathbb{R}$ , since given any $\varepsilon > 0$ there exists an $n \in \mathbb{N}$ such that $1/(n+1) < \varepsilon$, so $1 - \varepsilon < n/(n+1) < 1$, showing that $1$ is a limit point.

But if $x = n/(n+1)$ for some $n \in \mathbb{N}$ then we may take

$$\varepsilon = (n+1)/(n+2) - n/(n+1) = 1/(n+1)(n+2)$$

and then $(x - \varepsilon,\, x + \varepsilon) \cap A = \{n/(n+1)\}$, so $x$ is not a limit point of $A$. The upshot is that the set of limit points here is the singleton $\{1\}$.

(d) The only limit point of the set in Exercise 6.2 (d) is $0$.

6.19 Assume the result of Proposition 6.18, and first assume that the subset $A$ is closed in $X$. Then $\overline{A} = A$ by Proposition 6.11 (c). By Proposition 6.18 all limit points of $A$ are in $\overline{A}$, hence in $A$.

Conversely suppose that the subset $A \subseteq X$ contains all its limit point in $X$. Then by Proposition 6.18 $\overline{A} = A$ and $A$ is closed in $X$ by Proposition 6.11 (c).

6.21 (a) In each case we have already seen that the closure in $\mathbb{R}$ is $[a,\, b]$ (this is the same as Example 6.8 (a)) so it is enough to show that the interior is $(a,\, b)$. This follows from Proposition 6.21 (f) - the interior of a set $A$ in $X$ is the largest set contained in $A$ and open in $X$.

(b) We have seen that the closure of $\mathbb{Q}$ in $\mathbb{R}$ is $\mathbb{R}$. It is therefore enough to show that the interior of $\mathbb{Q}$ in $\mathbb{R}$ is $\emptyset$. But this is true, since given any $x \in \mathbb{R}$ and any open set $U$ in $\mathbb{R}$ containing $x$, there is an $\varepsilon > 0$ such that $(x - \varepsilon,\, x + \varepsilon) \subseteq U$. But any such open interval contains irrational numbers, so $U$ cannot be contained in $\mathbb{Q}$, so $x$ is not in the interior of $\mathbb{Q}$ in $\mathbb{R}$.

6.23 (a) By definition $\partial A = \overline{A} \setminus \mathring{A}$ , and $\mathring{A} \subseteq A \subseteq \overline{A}$ so $\mathring{A} = \overline{A} \setminus \partial A$. Since $\mathring{A} \subseteq A$, in fact $\mathring{A} = A \setminus \partial A$.

(b) This holds since for $x \in X$,

$x \in \overline{X \setminus A} \Leftrightarrow$ any open set $U \ni x$ has non-empty intersection with $X \setminus A \Leftrightarrow$ no open set $U \ni x$ is contained in $A \Leftrightarrow \quad x \notin \overset{\circ}{A} \quad \Leftrightarrow x \in X \setminus \overset{\circ}{A}$ .

(c) By definition $\partial A = \overline{A} \setminus \overset{\circ}{A}$ . Now using Exercise 2.1 (with the $C, D$ of that exercise taken to be $\overset{\circ}{A}$ , $\overline{A}$ respectively) we have $\overline{A} \setminus \overset{\circ}{A} = (X \setminus \overset{\circ}{A}\,) \cap \overline{A}$. So $\partial A = (X \setminus \overset{\circ}{A}\,) \cap \overline{A} = \overline{X \setminus A} \cap \overline{A}$, where the second equality uses (b) above. The second equality in the question follows by symmetry.

(d) This follows from (c), since $\partial A$ is the intersection of the two closed sets $\overline{X \setminus A}$ and $\overline{A}$.

6.25 Let the metrics on $X, Y$ be $d_X, d_Y$. First suppose that $f : X \to Y$ is continuous. Let $(x_n)$ be a sequence in $X$ converging to a point $x_0 \in X$. Then by continuity of $f$ at $x_0$, given $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x, y) < \delta$. Since $(x_n)$ converges to $x_0$ there exists an integer $N$ such that $d_X(x_n, x_0) < \delta$ whenever $n \geqslant N$. So for $n \geqslant N$ we have $d_Y(f(x_n), f(x_0)) < \varepsilon$. This proves that $(f(x_n))$ converges to $f(x_0)$.

Conversely suppose that $(f(x_n))$ converges to $f(x_0)$ whenever $(x_n)$ converges to a point $x_0$. We shall prove by contradiction that $f$ is continuous at $x_0$. For if it is not, then for some $\varepsilon > 0$ there is no $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$. In particular, for each $n \in \mathbb{N}$ there exists a point, call it $x_n$, such that $d_X(x, y) < 1/n$ yet $d_Y(f(x), f(y)) \geqslant \varepsilon$. Now $(x_n)$ converges to $x_0$ but $(f(x_n))$ does not converge to $f(x_0)$. This contradiction proves that $f$ is continuous at $x_0$, and the same applies at any point of $X$.

6.27 Since $d^{(2)}(x, y) \leqslant d(x, y)$ for all $x, y \in X$ , we see that as in Exercise 5.15 (a), $B^d_\varepsilon(x) \subseteq B^{(2)}_\varepsilon(x)$ for all $x \in X$ and $\varepsilon > 0$. Hence if $U \subseteq X$ is $d^{(2)}$ -open it is also $d$ -open. Conversely suppose that $U$ is $d$ -open and let $x \in U$. Then $B^d_\varepsilon(x) \subseteq U$ for some $\varepsilon > 0$, and we may take $\varepsilon < 1$. Then $B^{d^{(2)}}_\varepsilon(x) = B^d_\varepsilon(x) \subseteq U$, so $U$ is $d^{(2)}$ -open. This shows that $(X, d)$ and $(X, d^{(2)})$ are topologically equivalent.

Since $d^{(3)}(x, y) \leqslant d(x, y)$ for all $x, y \in X$, as for $d^{(2)}$ when a subset $U \subseteq X$ is $d^{(3)}$ -open it is also $d$ -open. Conversely suppose that $U$ is $d$ -open and $x \in U$. Then $B^d_\varepsilon(x) \subseteq U$ for some $\varepsilon > 0$. Let $\delta = \min\{\varepsilon/2, 1/2\}$. If $d^{(3)}(x, y) < \delta$ then $d^{(3)}(x, y) < 1/2$, so

$$d(x, y) = \frac{d^{(3)}(x, y)}{1 - d^{(3)}(x, y)} \leqslant 2d^{(3)}(x, y) < \varepsilon.$$

Then $B^{d^{(3)}}_\delta(x) \subseteq B^d_\varepsilon(x) \subseteq U$, so $U$ is $d^{(3)}$ -open. This proves that $(X, d)$ and $(X, d^{(3)})$ are topologically equivalent.

# Chapter 7

7.1 (a) Any topology on $\{0, 1\}$ has to contain $\emptyset$ and $\{0, 1\}$. It may contain either, neither or both of $\{0\}$ and $\{1\}$ and each of these possibilities gives a topology. So there are precisely four distinct topologies on $X = \{0, 1\}$, namely $\mathcal{T}_1 = \{X, \emptyset\}$, $\mathcal{T}_2 = \{X, \emptyset, \{0\}\}$, $\mathcal{T}_3 = \{X, \emptyset, \{1\}\}$ and $\mathcal{T}_4 = \{X, \emptyset, \{0\}, \{1\}\}$.

(b) The combinatorics of the situation rapidly increase in complexity with the number of points in $X$. If $X = \{0, 1, 2\}$ then again any topology on $X$ must contain $X$ and $\emptyset$, but there are now many topologies (29 in fact). One way to keep track of them is to list them by the number of singleton sets in them.

If there are no singleton sets in the topology, then the topology must have at most one set of order two in it (for if say $\{0, 1\}$ and $\{0, 2\}$ are in the topology then by the intersection property (T2) so is the singleton $\{0\}$ : we get four topologies $\{\emptyset, X\}$, $\{\emptyset, \{0, 1\}, X\}$, $\{\emptyset, \{0, 2\}, X\}$, $\{\emptyset, \{1, 2\}, X\}$.

If there is just one singleton set in the topology, and if it is $\{0\}$, then there are five possible topologies: these are $\{\emptyset, \{0\}, X\}$, $\{\emptyset, \{0\}, \{0, 1\}, X\}$, $\{\emptyset, \{0\}, \{0, 2\}, X\}$, $\{\emptyset, \{0\}, \{1, 2\}, X\}$, $\{\emptyset, \{0\}, \{0, 1\}, \{0, 2\}, X\}$.
There are five analagous topologies in which the only singleton is $\{1\}$ and another five in which the only singleton is $\{2\}$.

Next we list the topologies with precisely two singleton sets in them. There are those with just one set of order two and those with two sets of order two in them. For example if the two singleton sets are $\{0\}, \{1\}$ then we get just one topology with precisely one set of order two, namely $\{\emptyset, \{0, \}, \{1\}, \{0, 1\}, X\}$ . We get two topologies with these same singleton sets and precisly two sets of order two, $\{\emptyset, \{0\}, \{1\}, \{0, 1\}, \{0, 2\}, X\}$ and $\{\emptyset, \{0\}, \{1\}, \{0, 1\}, \{1, 2\}, X\}$. We also get three topologies in which the singletons are $\{0\}, \{2\}$ and three more in which the singletons are $\{1\}, \{2\}$ .

Finally, if the topology contains all three possible singletons, then it is the discrete topology (all subsets of $X$ are in the topology). Altogether this gives 29 distinct topologies.

7.3 Suppose that $\mathcal{T}_1, \mathcal{T}_2$ are topologies on a set $X$. Then so is $\mathcal{T}_1 \cap \mathcal{T}_2$. For
(T1) $\emptyset, X \in \mathcal{T}_1$ and $\emptyset, X \in \mathcal{T}_2$ so $\emptyset, X \in \mathcal{T}_1 \cap \mathcal{T}_2$.
(T2) If $U, V \in \mathcal{T}_1 \cap \mathcal{T}_2$ then $U, V \in \mathcal{T}_1$ and $\mathcal{T}_1$ is a topology so $U \cap V \in \mathcal{T}_1$. Similarly $U \cap V \in \mathcal{T}_2$ so $U \cap V \in \mathcal{T}_1 \cap \mathcal{T}_2$.
(T3) If $U_i \in \mathcal{T}_1 \cap \mathcal{T}_2$ for all $i$ in some index set $I$ , then for all $i \in I$, $U_i \in \mathcal{T}_1$ so $\bigcup_{i\in I} U_i \in \mathcal{T}_1$ since $\mathcal{T}_1$ is a topology. Similarly $\bigcup_{i\in I} \in \mathcal{T}_2$ so $\bigcup_{i\in I} U_1 \in \mathcal{T}_1 \cap \mathcal{T}_2$.

The union of the topologies $\{\emptyset, \{0\}, \{0, 1\}\}$ and $\{\emptyset, \{1\}, \{0, 1\}$ is the discrete topology on $\{0, 1\}$. But the union of the topologies $\{\emptyset, \{0\}, X\}$ and $\{\emptyset, \{1\}, X\}$ on $X = \{0, 1, 2\}$ is $\{\emptyset, \{0\}, \{1\}, X\}$ which is not a topology on $X$ since the union of $\{0\}, \{1\}$ is not in it.

We see that the argument above for the intersection of two topologies works exactly the same way for the intersection of any family of topologies.

7.5 Let $\mathcal{T}$ be defined on a set $X$ as in Example 7.9.
(T1) By definition $\emptyset \in \mathcal{T}$. Also, since $X \setminus X = \emptyset$ is finite, $X \in \mathcal{T}$.
(T2) Suppose that $U, V \in \mathcal{T}$. If either is empty, then so is $U \cap V$ so $U \cap V \in \mathcal{T}$. Otherwise $X \setminus U$ and $X \setminus V$ are both finite, hence $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$ is also finite, so $U \cap V \in \mathcal{T}$.
(T3) Suppose that $U_i \in \mathcal{T}$ for all $i$ in some index set $I$. If all the $U_i$ are empty then so is $\bigcup_{i\in I} U_i$ and hence it is in $\mathcal{T}$. Otherwise $U_{i_0} \neq \emptyset$ for some $i_0 \in I$, so $X \setminus U_{i_0}$ is finite. Then $X \setminus \left(\bigcup_{i\in I} U_i\right) \subseteq X \setminus U_{i_0}$ is also finite, so $\bigcup_{i\in I} U_i \in \mathcal{T}$.

# Chapter 8

8.1 (a) In this case the inverse image of any open set is itself hence it is open, so $f$ is continuous.

(b) Let the constant value of $f$ be $y_0 \in Y$. Then $f^{-1}(U) = X$ if $y_0 \in U$ and $f^{-1}(U) = \emptyset$ if $y_0 \notin U$. In either case $f^{-1}(U)$ is open so $f$ is continuous.

(c) In this case $f^{-1}(U)$ is open in $X$ for any subset $U \subseteq Y$ so $f$ is continuous.

(d) The only open sets of $Y$ are $\emptyset, Y$. Now $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$. So $f$ is continuous since both $\emptyset$ and $X$ are open in $X$.

8.3 Suppose first that $A$ is open in $X$ . The only open sets in $\mathbb{S}$ are $\emptyset, \mathbb{S}$ and $\{1\}$. Now $\chi_A^{-1}(\emptyset) = \emptyset, \chi_A^{-1}(\mathbb{S}) = X$ and $\chi_A^{-1}(1) = A$, all of which are open in $X$ so $\chi_A$ is continuous.

Conversely suppose that $\chi_A$ is continuous. Then $A$ is open in $X$ since $A = \chi_A^{-1}(1)$ and $\{1\}$ is open in $\mathbb{S}$.

8.5 We need to show that any open subset $U \subseteq \mathbb{R}$ is a union of finite open intervals. By definition of the usual topology on $\mathbb{R}$, for any $x \in U$ there is some $\varepsilon_x > 0$ such that $(x - \varepsilon_x,\, x + \varepsilon_x) \subseteq U$. It is straightforward to check that

$$U = \bigcup_{x \in U} (x - \varepsilon_x,\, x + \varepsilon_x).$$

8.7 We have to show that $\mathcal{B}$ is a basis, and that it is countable. We first show that $\mathcal{B}$ is a basis. For this we need to show that any open subset $U \subseteq \mathbb{R}^2$ is the union of a subfamily of $\mathcal{B}$.

So let $U$ be an open subset of $\mathbb{R}^2$ and let $(x,\, y) \in U$. It is enough to show that there is a set $B \in \mathcal{B}$ such that $(x,\, y) \in B \subseteq U$. First, there exists $\varepsilon > 0$ such that $B_{3\varepsilon}((x,\, y)) \subseteq U$. Now choose a rational number $q$ such that $\varepsilon < q < 2\varepsilon$. Let $q_1,\, q_2$ be rational numbers with $|x - q_1| < \varepsilon/\sqrt{2}$ and $|y - q_2| < \varepsilon/\sqrt{2}$. Let us write $d$ for the Euclidean distance in $\mathbb{R}^2$. Then

$$d((q_1,\, q_2),\, (x,\, y)) = \sqrt{(x - q_1)^2 + (y - q_2)^2} < \varepsilon.$$

Hence $(x,\, y) \in B_\varepsilon((q_1,\, q_2)) \subseteq B_q((q_1,\, q_2)) \in \mathcal{B}$. Also,

$$B_q((q_1,\, q_2)) \subseteq B_{3\varepsilon}((x,\, y)) \subseteq U: \text{ for if } (x',\, y') \in B_q(q_1,\, q_2) \text{ then}$$

$$d((x',\, y'),\, (x,\, y)) \leqslant d((x',\, y'),\, (q_1,\, q_1)) + d((q_1,\, q_2),\, (x,\, y)) < q + \varepsilon < 3\varepsilon.$$

This shows that $U$ is a union of sets in $\mathcal{B}$.

To show that $\mathcal{B}$ is countable, note that there is an injective function from $\mathcal{B}$ to $\mathbb{Q}^3$ defined by $B_q((q_1,\, q_2)) \mapsto (q,\, q_1,\, q_2)$. Now $\mathcal{B}$ is countable by standard facts about countable sets: $\mathbb{Q}$ is countable, a finite product of countable sets is countable, and any set from which there is an injective function to a countable set is countable.

# Chapter 9

9.1 The complement of any subset $V$ of a discrete space $X$ is open in $X$, so $V$ is closed in $X$.

9.3 We may choose for example $U = (0,\, 1) \cup (2,\, 4)$, $V = (1,\, 3)$. Then

$$U \cap \overline{V} = (2,\, 3],\ \ \overline{U} \cap V = [2,\, 3),\ \ \overline{U} \cap \overline{V} = \{1\} \cup [2,\, 3],\ \ \overline{U \cap V} = [2,\, 3].$$

9.5 (a) This is false in general. For a counterexample let $X$ be the space $\{0,\, 1\}$ with the discrete topology, let $Y$ be the space $\{0,\, 1\}$ with the indiscrete topology, and let $f$ be the identity function. Then $f$ is continuous e.g. by Exercise 8.1 (c) or (d). Also, $A = \{0\}$ is closed in $X$ but $f(A) = A$ is not closed in $Y$. (The analogous counterexample would work for any set with at least two points in it.)

(b) Again this is false: Exercise 9.3 gives a counterexample - take $A = (0,\, 1) \cup (2,\, 4)$ and $B = (1,\, 3)$ in $\mathbb{R}$, and we have $A \cap \overline{B} = (2,\, 3]$ while $\overline{A \cap B} = [2,\, 3]$.

(c) This is also false. Let $f$ be as in the counterexample to (a) and let $B = \{0\}$. Then the closure of $B$ in $Y$ is $\overline{B} = \{0,\, 1\}$ so $f^{-1}(\overline{B}) = \{0,\, 1\}$ but $f^{-1}(B) = \{0\}$ so the closure of this in $X$ is $\overline{f^{-1}(B)} = \{0\}$.

9.7 Suppose first that $f : X \to Y$ is continuous and that $A \subseteq X$. Let $y \in f(\overline{A})$, say $y = f(x)$ where $x \in \overline{A}$. Let $U$ be any open subset of $Y$ containing $y$. Then $f^{-1}(U)$ is open in $X$ and $x \in f^{-1}(U)$. Hence there exists $a \in A \cap f^{-1}(U)$, and then $f(a) \in U$. Hence $y \in \overline{f(A)}$. This shows that $f(\overline{A}) \subseteq \overline{f(A)}$.

Conversely suppose that $f(\overline{A}) \subseteq \overline{f(A)}$ for any subset $A \subseteq X$. In particular we apply this when $A = f^{-1}(V)$ where $V$ is closed in $Y$. Then

$$f(\overline{f^{-1}(V)} \subseteq \overline{f(f^{-1}(V))} \subseteq \overline{V} = V.$$

Hence $\overline{f^{-1}(V)} \subseteq f^{-1}(V)$. Since $f^{-1}(V) \subseteq \overline{f^{-1}(V)}$ we have $\overline{f^{-1}(V)} = f^{-1}(V)$ so by Proposition 9.10 (c) $f^{-1}(V)$ is closed in $X$, showing that $f$ is continuous.

9.9 (a) If $a \in \overset{\circ}{A}$ then by definition there is some open set $U$ of $X$ such that $a \in U \subset A$. In particular then $a \in A$. So $\overset{\circ}{A} \subseteq A$.

(b) if $A \subseteq B$ and $x \in \overset{\circ}{A}$ then by definition there is some open subset $U$ of $X$ such that $x \in U \subseteq A$. Since $A \subseteq B$ then also $U \subseteq B$, so $x \in \overset{\circ}{B}$. This proves that $\overset{\circ}{A} \subseteq \overset{\circ}{B}$.

(c) If $A$ is open in $X$ then for every $a \in A$ there is an open set $U$ (namely $U = A$ ) such that $a \in U \subseteq A$, so $a \in \overset{\circ}{A}$. This shows that $A \subseteq \overset{\circ}{A}$ and together with (a) we get $A = \overset{\circ}{A}$.

Conversely if $A = \overset{\circ}{A}$ then for every $a \in A$ there exists an open set, call it $U_a$, such that $a \in U_a \subseteq A$. It is straightforward to check that $A = \bigcup\limits_{a \in A} U_a$ which is a union of sets open in $X$ hence is open in $X$.

(d) by (a) the interior of $\overset{\circ}{A}$ is contained in $\overset{\circ}{A}$. Conversely suppose that $a \in \overset{\circ}{A}$. Then there exists a subset $U$ open in $X$ such that $a \in U \subseteq A$. Now for any point $x \in U$ we have $x \in U \subseteq A$, so also $x \in \overset{\circ}{A}$. This shows that $a \in U \subseteq \overset{\circ}{A}$, and $U$ is open in $X$, so $a$ is in the interior of $\overset{\circ}{A}$. These together show that the interior of $\overset{\circ}{A}$ is $\overset{\circ}{A}$.

(e) This follows from (c) and (d).

(f) We know that $\overset{\circ}{A}$ is open in $X$ from (e). Suppose that $B$ is open in $X$ and that $B \subseteq A$. By (b) then $\overset{\circ}{B} \subseteq \overset{\circ}{A}$. Since $B$ is open we have $\overset{\circ}{B} = B$ by (c). So $B \subseteq \overset{\circ}{A}$, which says that $\overset{\circ}{A}$ is the largest open subset of $X$ contained in $A$.

9.11 Since $\overset{\circ}{A_i} \subseteq A_i$ for each $i = 1, 2, \ldots, m$ we get $\bigcap_{i=1}^{m} \overset{\circ}{A_i} \subseteq \bigcap_{i=1}^{m} A_i$. Also, $\bigcap_{i=1}^{m} \overset{\circ}{A_i}$ is the intersection of a finite family of open sets hence is open in $X$, hence it is contained in the interior of $\bigcap_{i=1}^{m} A_i$. Conversely $\bigcap_{i=1}^{m} A_i \subseteq A_j$ for each $j = 1, 2, \ldots, m$; it follows (9.9 (b)) that the interior of $\bigcap_{i=1}^{m} A_i$ is contained in $\overset{\circ}{A_j}$ for each $j = 1, 2, \ldots, m$, so the interior of $\bigcap_{i=1}^{m} A_i$ is contained in $\bigcap_{i=1}^{m} \overset{\circ}{A_i}$. This proves the result.

9.13 This follows from the fact that $\partial A = \overline{A} \cap \overline{X \setminus A}$ (Proposition 9.20) since each of $\overline{A}$, $\overline{X \setminus A}$ is closed in $X$ hence so is their intersection.

9.15 Since $\partial A = \overline{A} \setminus \overset{\circ}{A}$, we have $\partial A \cap \overset{\circ}{A} = \emptyset$. By the definition $\partial A = \overline{A} \setminus \overset{\circ}{A}$ we know that $\partial A \subseteq \overline{A}$ and $\overset{\circ}{A} \subseteq A \subseteq \overline{A}$. So the disjoint union $\partial A \sqcup A \subseteq \overline{A}$. Conversely since $\partial A = \overline{A} \setminus \overset{\circ}{A}$, we have $\overline{A} \subseteq \overset{\circ}{A} \sqcup \partial A$. These two together show that $\overline{A} = \overset{\circ}{A} \sqcup \partial A$.

Now if $B \subseteq X$ and $B \cap A \neq \emptyset$ then $B \cap \overline{A} \neq \emptyset$ so either $B \cap \overset{\circ}{A} \neq \emptyset$ or $B \cap \partial A \neq \emptyset$.

# Chapter 10

10.1 The subspace topology $\mathcal{T}_A$ consists of all sets $U \cap A$ where $U \in \mathcal{T}$. Hence $\mathcal{T}_A = \{\emptyset, \{a\}, A\}$.

10.3 We have to show that the subspace topology $\mathcal{T}_A$ on $A$ is the same as the co-finite topology on $A$. First suppose that $V \subseteq A$ is in the co-finite topology for

$A$. Either $V = \emptyset$ and then $V = A \cap \emptyset \in \mathcal{T}_A$, or $A \setminus V$ is finite. In this latter case let $U = (X \setminus A) \cup V$. Then $A \cap U = V$, and $U$ is in the co-finite topology for $X$ since $X \setminus U = A \setminus V$ and the latter is finite.

Conversely suppose that $V = A \cap U$ where $U$ is in the co-finite topology $\mathcal{T}$ for $X$. Then either $U = \emptyset$, so $V = A \cap U = \emptyset$, and $V$ is in the co-finite topology for $A$ , or else $X \setminus U$ is finite, in which case $A \setminus V \subseteq X \setminus U$ is finite and again $V$ is in the co-finite topology for $A$.

10.5 Since $V$ is closed in $X$ its complement $X \setminus V$ is open in $X$. Now we have $A \setminus (V \cap A) = A \cap (X \setminus V)$ by Exercise 2.2. So $A \setminus (V \cap A) \in \mathcal{T}_A$. This shows that $V \cap A$ is closed in $(A, \mathcal{T}_A)$.

10.7 (a) We use Proposition 3.13: for any subset $B \subseteq Y$ we have

$$f^{-1}(B) = \bigcup_{i \in I} (f|U_i)^{-1}(B).$$

Now let $B$ be open in $Y$. For each $i \in I$ continuity of $f|U_i$ implies that $(f|U_i)^{-1}(B)$ is open in $U_i$ and hence, by Exercise 10.6 (a), open in $X$. Hence $f^{-1}(B)$ is a union of sets open in $X$ , so $f^{-1}(B)$ is open in $X$. Thus $f$ is continuous as required.

(b) We again use Proposition 3.13: for any subset $B \subseteq Y$ we have

$$f^{-1}(B) = \bigcup_{i \in I} (f|V_i)^{-1}(B).$$

Now suppose that $B$ is closed in $Y$. Then continuity of $f|V_i$ ensures that $(f|V_i)^{-1}(B)$ is closed in $V$ , and hence, by Exercise 10.6 (b) it is closed in $X$. Hence $f^{-1}(B)$ is the union of a finite number of sets closed in $X$, so $f^{-1}(B)$ is closed in $X$, and $f$ is continuous as required.

10.9 (a) First suppose $x \in B_1$. Then $x \in X_1$ and also for any set $W$ open in $X_1$ with $x \in W$ we have $W \cap A \neq \emptyset$. Now let $U$ be any set open in $X_2$ with $x \in U$. Then $W = U \cap X_1$ is open in $X_1$ and contains $x$, so $W \cap A \neq \emptyset$. Then $U \cap A = U \cap (A \cap X_1) = (U \cap X_1) \cap A = W \cap A \neq \emptyset$, so $x \in B_2$. Since also $x \in X_1$ this shows that $B_1 \subseteq B_2 \cap X_1$.

Conversely suppose that $x \in B_2 \cap X_1$. Then $x \in X_1$ and for any subset $U$ open in $X_2$ we know $U \cap A \neq \emptyset$. Now let $W$ be an open subset of $X_1$ with $x \in W$. Then $W = X_1 \cap U$ for some $U$ open in $X_2$ with $x \in U$. Hence $U \cap A \neq \emptyset$, so since $A \subseteq X_1$ we have $U \cap A = U \cap X_1 \cap A = W \cap A$, so $W \cap A \neq \emptyset$ , showing that $x \in B_1$.

Taking these two together, we have $B_1 = B_2 \cap X_1$.

10.11 Any singleton set $\{(x, y)\}$ in $X \times Y$ is the product $\{x\} \times \{y\}$ of sets which are open in $X, Y$ since they have the discrete topology so $\{(x, y)\}$ is open in the product topology. Hence any subset of $X \times Y$ is open in the product topology, which is therefore discrete.

10.13 Consider the case when $Y$ is infinite, and $X$ contains at least two points. Then we may let $U$ be a non-empty open subset of $X$ with $U \neq X$. The complement of $U \times Y$ is $(X \setminus U) \times Y$, which is infinite. So $U \times Y$ is not open in the co-finite topology on $X \times Y$ although it is open in the product of the co-finite topologies on $X$ and $Y$.

10.15 (a) Any open subset $W$ of $X \times Y$ is a union $\bigcup_{i \in I} U_i \times V_i$ for some index set $I$ , where each $U_i$ is open in $X$ and each $V_i$ is open in $Y$. We may as well assume that no $V_i$ (and no $U_i$ ) is empty, since if it were then $U_i \times V_i$ would be empty, and hence does not contribute to the union. The point of this is that $p_X(U_i \times V_i) = U_i$ for all $i \in I$. Now

$$p_X(W) = p_X\left(\bigcup_{i \in I} U_i \times V_i\right) = \bigcup_{i \in I} p_X(U_i \times V_i) = \bigcup_{i \in I} U_i,$$

which is open in $X$ as a union of open sets. Similarly $p_Y(W)$ is open in $Y$.

(b) Consider the set $W = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$ . This is closed in $\mathbb{R}^2$ : a painless way to see this is to consider the function $m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $m(x, y) = xy$. Since $m$ is continuous by Propositions 8.3 and 5.17, and $\{1\}$ is closed in $\mathbb{R}$, $W = m^{-1}(1)$ is closed in $\mathbb{R} \times \mathbb{R}$ by Proposition 9.5. But $p_1(W) = \mathbb{R} \setminus \{0\}$ is not closed in $\mathbb{R}$.

10.17 Since $t$ is clearly one-one onto, it is enough to prove that $t$ and $t^{-1}$ are continuous. Now $t$ is continuous by Proposition 10.11, since if $p_1, p_2$ are the projections of $X \times X$ on the first, second factors, then $p_1 \circ t = p_2$ and $p_2 \circ t = p_1$ and $p_1, p_2$ are both continuous. Now we observe that $t$ is self-inverse, i.e. $t^{-1} = t$ so $t$ is a homeomorphism.

10.19 (a) The graph of $f$ is a curve through the point $(0, 1)$ which has the lines $x = -1$, $x = 1$ as asymptotes. We argue as in the proof of Proposition 10.18: let $\theta : X \to G_f$ be defined by $\theta(x) = (x, f(x))$ and let $\phi : G_f \to X$ be defined by $\phi(x, f(x)) = x$. Then $\theta$ and $\phi$ are easily seen to be mutually inverse. Continuity of $\theta$ follows from Proposition 10.11, since $p_1 \circ \theta$ is the identity map of $X$ and $p_2 \circ \theta$ is the continuous function $f$. Continuity of $\phi$ follows since $\phi$ is the restriction to $G_f$ of the continuous projection $p_1 : X \times \mathbb{R} \to X$. Hence $\theta$ is a homeomorphism (with inverse $\phi$ ).

(b) The graph of $f$ is not easy to draw, but it oscillates up and down with decreasing amplitude as $x$ approaches $0$ from the right. The continuity of $f : [0, \infty) \to \mathbb{R}$ on $(0, \infty)$ follows from continuity of the sine function together with Propositions 8.3 and 5.17. Continuity (from the right) at $0$ follows from Exercise 4.14. Now arguing as in Proposition 10.18 we see that $x \mapsto (x, f(x))$ defines a homeomorphism from $[0, \infty)$ to $G_f$.

# Chapter 11

11.1 Suppose that $x, y$ are distinct point in a space $X$ with the indiscrete topology. Then there are no disjoint open sets $U, V$ with $x \in U, y \in V$ since the only open set containing $x$ is $X$ , which also contains $y$.

11.3 We can prove this by induction on $n$. When $n=2$ the conclusion is simply the Hausdorff condition. Suppose the result is true for a given integer $n$ with $n \geqslant 2$ and let $x_1, x_2, \ldots, x_{n+1}$ be distinct points of $X$. By inductive hypothesis, for $i = 1, 2, \ldots, n$ there exist pairwise disjoint open sets $W_i$ with $x_i \in W_i$. Also, by the Hausdorff condtion, for each $i = 1, 2, \ldots, n$ there exist disjoint open sets $S_i, T_i$ such that $x_i \in S_i, x_{n+1} \in T_i$. For each $i = 1, 2, \ldots, n$ put $U_i = S_i \cap W_i$, and put $U_{n+1} = \bigcap_{i=1}^{n} T_i$. Then $U_1, U_2, \ldots, U_n$ are disjoint since $W_1, W_2, \ldots, W_n$ are, and for each $i = 1, 2, \ldots, n$ we have $U_i \cap U_{n+1} = \emptyset$ since $U_i \subseteq S_i$ and $U_{n+1} \subseteq T_i$. Also, by construction each of $U_1, U_2, \ldots U_n, U_{n+1}$ is open in $X$. Thus $U_1, U_2, \ldots, U_{n+1}$ are pairwise disjoint open sets. This completes the inductive step.

11.5 We prove that $(X \times Y) \setminus G_f$ is open in $X \times Y$. (Once we have opted for this, the rest of the proof 'follows its nose'.) It is enough, by Proposition 7.2, to show that for every point $(x, y) \in (X \times Y) \setminus G_f$ there exists an open subset $W$ of $X \times Y$ with $(x, y) \in W \subseteq (X \times Y) \setminus G_f$. So let $(x, y) \in (X \times Y) \setminus G_f$. Then $(x, y) \notin G_f$ so $f(x) \neq y$. Since $Y$ is Hausdorff there exist disjoint open sets $V_1, V$ of $Y$ such that $f(x) \in V_1, y \in V$. Since $f$ is continuous, $U = f^{-1}(V_1)$ is open in $X$. Note that $x \in U$ since $f(x) \in V_1$. Also, $y \in V$. Write $W = U \times V$. Then $(x, y) \in U \times V = W$, and $W \subseteq (X \times Y) \setminus G_f$ since if $(x', y') \in W$ then $x' \in U$ and $y' \in V$ so $f(x') \in V_1$ and $y' \in V$, but $V_1 \cap V = \emptyset$ so $f(x') \neq y'$, which says that $(x', y') \notin G_f$.

11.7 (a) Suppose first that $X$ is a Hausdorff space. We shall prove that $(X \times X) \setminus \Delta$ is open in $X \times X$ from which it will follow that $\Delta$ is closed in $X \times X$. (This proof is almost identical to that in Exercise 11.5.) So let $(x, y) \in (X \times X) \setminus \Delta$. By Proposition 7.2 it is enough to show that there is an open set $W$ of $X \times X$ with $(x, y) \in W \subseteq (X \times X) \setminus \Delta$. Since $(x, y) \in (X \times X) \setminus \Delta$ we have $(x, y) \notin \Delta$, so $y \neq x$. Since $X$ is Hausdorff there exist disjoint open subsets $U, V$ of $X$ such that $x \in U, y \in V$. Then $W = U \times V$ is an open subset of $X \times X$, and $(x, y) \in U \times V$. Moreover $W \subseteq (X \times X) \setminus \Delta$ since if $(x', y') \in W$ then $x' \in U, y' \in V$ and $U \cap V = \emptyset$ so $y' \neq x'$, which says $(x', y') \notin \Delta$.

Conversely suppose that $\Delta$ is closed in $X \times X$. Then $(X \times X) \setminus \Delta$ is open in $X \times X$. Let $x, y$ be distinct points of $X$. Then $y \neq x$ so $(x, y) \notin \Delta$. Hence $(x, y)$ is in the open set $(X \times X) \setminus \Delta$ and by definition of the product topology there exist open sets $U, V$ of $X$ such that $(x, y) \in U \times V \subseteq (X \times X) \setminus \Delta$. Now $x \in U, y \in V$ and $U \cap V = \emptyset$ since if $z \in U \cap V$ then $(z, z) \in \Delta \cap (U \cap V) = \emptyset$. So $X$ is Hausdorff.

(b) Consider the characteristic function $\chi_A : X \times X \to \mathbb{S}$ of the set $A = (X \times X) \setminus \Delta$. Since $\chi_A^{-1}(\emptyset) = \emptyset, \chi_A^{-1}(1) = A$ and $\chi_A^{-1}\mathbb{S} = X \times X$ we have that $\chi_A$ is continuous iff $A = (X \times X) \setminus \Delta$ is open in $X \times X$, i.e. iff $\Delta$ is closed in $X \times X$, and by (a) this holds iff $X$ is Hausdorff.

11.9 Since $f_A$ and $f_B$ are continuous, so is $g$. So since $(-\infty, 0)$ and $(0, \infty)$ are open in $\mathbb{R}$ we know that $g^{-1}(-\infty, 0)$ and $g^{-1}(0, \infty)$ are open in $X$. Also, $A$ and $B$ are closed sets, so $\overline{A} = A$ and $\overline{B} = B$. Now from Exercise 6.16, $f_A(x) \geqslant 0$ for all $x \in X$ and $f_A(x) = 0$ iff $x \in A$. Similarly $f_B(x) \geqslant 0$ for all $x \in X$ and $f_B(x) = 0$ iff $x \in B$. Hence, since $A$ and $B$ are disjoint, for $x \in A$ we have $f_A(x) = 0$ and $f_B(x) > 0$, so $g(x) < 0$. Similarly for $x \in B$ we have $g(x) > 0$. Thus $A \subseteq g^{-1}(-\infty, 0)$ and $B \subseteq g^{-1}(0, \infty)$. Finally, $g^{-1}(-\infty, 0)$

and $g^{-1}(0, \infty)$ are clearly disjoint (if $x \in g^{-1}(-\infty, 0)$ then $g(x) < 0$, while if $x \in g^{-1}(0, \infty)$ then $g(x) > 0$. ).

# Chapter 12

12.1 (i) has the partition $\{B_1((1, 0)), B_1((-1, 0))\}$ so it is not connected, hence not path-connected. The others are all path-connected and hence connected. We can see this for (ii) and (iii) by observing that any point in $\overline{B_1((1, 0))}$ may be connected by a straight-line segment to the centre $(1, 0)$ (and this segment lies entirely in $\overline{B_1((1, 0))}$ ), any point in $X = \overline{B_1((-1, 0))}$ or $B_1((-1, 0))$ can be connected by a straight-line segment in $X$ (or in $B_1((-1, 0))$ ) to the centre $(-1, 0)$. Moreover the points $(1, 0)$ and $(-1, 0)$ can be connected by straight-line segment which lies entirely in $\overline{B_1((-1, 0))} \cup B_1((1, 0))$, hence certainly in $\overline{B_1((-1, 0))} \cup \overline{B_1((1, 0))}$.

To see that (iv) is path-connected note that any point $(q, y) \in \mathbb{Q} \times [0, 1]$ can be connected by a straight-line segment within $\mathbb{Q} \times [0, 1]$ to the point $(q, 1)$ and that any two points in the line $\mathbb{R} \times \{1\}$ can be connected by a straight-line segment entirely within this line.

Finally to see that the set $S$ in (v) is path-connected, it is enough to show that any point $(x, y) \in S$ can be connected to the origin $(0, 0)$ by a path in $S$ . If $x \in \mathbb{Q}$ we first connect $(x, y)$ to $(x, 0)$ by a vertical line-segment in $S$ , then we connect $(x, 0)$ to $(0, 0)$ by a (horizontal) line-segment in $S$. If $x \notin \mathbb{Q}$ then $y \in \mathbb{Q}$ and we first connect $(x, y)$ to $(0, y)$ by a straight-line segment in $S$ and then we connect $(0, y)$ to $(0, 0)$ by a (vertical) straight-line segment in $S$.

12.3 Suppose that $X$ is an infinite set with the co-finite topology and let $U, V$ be non-empty open sets in $X$ . The $X \setminus U$ and $X \setminus V$ are both finite, hence $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$ is finite. This shows that $U \cap V$ is non-empty - indeed it is infinite. Hence there is no partition of $X$, so $X$ is connected.

12.5 We may prove that this is true for any finite integer $n$ by induction. The result is certainly true when $n = 1$ and if it holds for a given integer $n$ then for $n + 1$ it follows from Proposition 12.16 applied to the connected sets $A_1 \cup A_2 \cup \ldots \cup A_n$ (which is connected by inductive hypothesis) and $A_{n+1}$ : these have non-empty intersection since $A_n \cap A_{n+1} \neq \emptyset$.

The analogous result is also true for an infinite sequence $(A_i)$ of connected sets such that $A_i \cap A_{i+1} \neq \emptyset$ for each positive integer $i$ . For suppose that $\{U, V\}$ were a partition of $\displaystyle\bigcup_{i=1}^{\infty} A_i$. For each $i \in \mathbb{N}$ we have either $A_i \subseteq U$ or $A_i \subseteq V$, since otherwise $\{U \cap A_i, V \cap A_i\}$ would be a partition of $A_i$. Let $I_U$ be the subset of all $i \in \mathbb{N}$ for which $A_i \subseteq U$ and let $I_V$ be the analogous set for $V$. Suppose w.l.o.g. that $1 \in I_U$ (otherwise switch the names of $U$ and $V$ ). Now $n \in I_U$ implies $n + 1 \in I_U$, for if $A_n \subseteq U$ then the connected set $A_n \cup A_{n+1}$ is also contained in $U$, otherwise $\{(A_n \cup A_{n+1}) \cap U, (A_n \cup A_{n+1}) \cap V\}$ would partition $A_n \cup A_{n+1}$. It follows that $\mathbb{N} \subseteq I_U$, so $\displaystyle\bigcup_{i=1}^{\infty} A_i \subseteq U$ , contradicting the assumption that $\{U, V\}$ is a partition of this union.

12.7 This follows from the intermediate value theorem since we may show that if the polynomial function is written $f$, then $f(x)$ takes a different sign for $x$ large and negative from its sign for $x$ large and positive, so its graph must cross the $x$ -axis somewhere. Explicitly, we may as well assume that the polynomial is monic (has leading coefficient 1)

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = x^n + g(x), \text{ say.}$$

Since $g(x)/x^n \to 0$ as $x \to \pm\infty$, there exists $\Delta \in \mathbb{R}$ such that $|g(x)/x^n| < 1$ for $|x| \geqslant \Delta$. This says that for large $|x|$ , we have that $f(x)$ and $x^n$ have the same sign. But $n$ is odd, so $f(x) < 0$ for $x$ large and negative, and $f(x) > 0$ for $x$ large and positive.

12.9 Define $g$ as the hint suggests. Then $g$ is continuous on $[0, 1]$ and

$$\begin{aligned} g(0)+g(1/n)+\ldots+g((n-1)/n) &= f(0)-f(1/n)+f(1/n)+\ldots+f((n-1)/n)-f(1) \\ &= f(0) - f(1) = 0. \end{aligned}$$

Hence we have only the following two cases to consider.
Case 1. All the $g(i/n)$ are zero. Now if $g(i/n) = 0$ then $f(i/n) = f((i+1)/n)$ and the conclusion holds with $x = i/n$.

Case 2, For some $i = 0, 1, \ldots (n-1)$ the values $g(i/n), g((i+1)/n$ have opposite signs. Then by the intermediate value theorem there exists some $x \in (i/n, (i+1)/n)$ with $g(x) = 0$, so $f(x) = f(x + 1/n)$.

12.11 (a) This is false. For example let $X = Y = \mathbb{R}$ and $A = B = \{0\}$. Then

$$X \setminus A = Y \setminus B = \mathbb{R} \setminus \{0\}$$

which is not connected, but $X \times Y \setminus (A \times B) = \mathbb{R}^2 \setminus \{(0, \mathbf{0})\}$ which is path-connected and hence connected.

Note that common sense suggests (b) false (c) true, since the conclusion is the same for both, but the hypotheses are stronger in (c). (This proves *nothing*, but it is suggestive.)

(b) This is false. For example let $X = \mathbb{R}$ and let $A = \{0, 1\}, B = (0, 1]$. Then both of $A \cap B = \{1\}$ and $A \cup B = [0, 1]$ are connected, but $A$ is not connected.

(c) This is true. We prove it in the style of Definition 12.1. So let $f : A \to \{0, 1\}$ be continuous, where $\{0, 1\}$ has the discrete topology. Then $f|A \cap B$ is continuous, and since $A \cap B$ is connected, $f|A \cap B$ is constant, say with value $c$ (where $c = 0$ or $1$ ). Define $g : A \cup B \to \{0, 1\}$ by $g|A = f, g|B = c$. Then $g$ is continuous by Exercise !0.7 (b), since each of $A$, $B$ is closed in $X$ and hence in $A \cup B$, and on the intersection $A \cap B$ the two definitions agree. But $A \cup B$ is connected, so $g$ is constant. In particular this implies that $f : A \to \{0, 1\}$ is constant, so $X$ is connected. Similarly $B$ is connected.

12.13 Let $y_1, y_2 \in Y$. Since $f$ is onto, $f(x_1) = y_1, f(x_2) = y_2$ for some $x_1, x_2 \in X$. Let $g : [0, 1] \to X$ be a continuous path in $X$ from $x_1$ to $x_2$. Then $f \circ g : [0, 1] \to Y$ is a continuous path in $Y$ from $y_1$ to $y_2$. So $Y$ is path-connected.

12.15 This follows from Exercise 9.14 (b), which says that a subset $A$ is open and closed in $X$ iff it has an empty boundary. Since $X$ is connected iff no proper (i.e

$\neq X$ ) non-empty subset of $X$ is open and closed in $X$ , it is connecetd iff every proper non-empty subset of $X$ has non-empty boundary.

12.17 Suppose that $f : A \cup B \to \{0, 1\}$ is continuous, where $\{0, 1\}$ has the discrete topology. Since $A$ and $B$ are connected, both $f|A$ and $f|B$ are constant, say with values $c_A$ and $c_B$ in $\{0, 1\}$. Let $a \in A \cap \overline{B}$. Then $f^{-1}(c_A)$ is an open set containing $a$ and hence some point $b \in B$. But then $f(b) = c_A$ and also $f(b) = c_B$ since $b \in B$. So $c_A = c_B$, and $f$ is constant. Hence $A \cup B$ is connected.

12.19 The idea of this example is an infinite ladder where we kick away a rung at a time. Explicitly, let

$$V_n = ([0, \infty) \times \{0, 1\}) \cup \bigcup_{i \in \mathbb{N},\, i \geqslant n} \{i\} \times [0, 1].$$

Then it is clear that each $V_n$ is path-connecetd hence connected and that $V_n \supseteq V_{n+1}$ for each $n \in \mathbb{N}$. But

$$\bigcap_{n=1}^{\infty} V_n = [0\, \infty) \times \{0, 1\}, \quad \text{which is not connected.}$$

# Chapter 13

13.1 Suppose that the space $X$ has the indiscrete topology. Then the only open sets in $X$ are $\emptyset$, $X$. So any open cover of $X$ must contain the set $X$, and $\{X\}$ is a finite subcover.

13.3 Suppose that $\mathcal{U}$ is an open cover of $A \cup B$. In particular $\mathcal{U}$ is an open cover of $A$, so there is a finite subfamily $\mathcal{U}_A$ of $\mathcal{U}$ which covers $A$. Similarly there is a finite subfamily $\mathcal{U}_B$ of $\mathcal{U}$ which covers $B$. Then $\mathcal{U}_A \cup \mathcal{U}_B$ is a finite subcover of $\mathcal{U}$ which covers $A \cup B$ and this proves that $A \cup B$ is compact.

13.5 Suppose that $\mathcal{U}$ is any open cover of $(X, \mathcal{T})$. Since $\mathcal{T} \subseteq \mathcal{T}'$, each set in $\mathcal{U}$ is in $\mathcal{T}'$, hence $\mathcal{U}$ is an open cover of $(X, \mathcal{T}')$ as well. But $(X, \mathcal{T}')$ is compact, so there is a finite subcover. This proves that $(X, \mathcal{T})$ is compact.

13.7 This is immediate since any finite subset of a space is compact.

13.9 Suppose first that $X \subseteq \mathbb{R}$ is unbounded. Let us define $f : X \to \mathbb{R}$ by $f(x) = 1/(1 + |x|)$. Then the lower bound of $f$ is $0$, since $f(x) > 0$ for all $x \in X$, but for any $\delta > 0$ there exists $x \in X$ such that $1 + |x| > 1/\delta$, and then $f(x) < \delta$. But $f$ does not attain its lower bound $0$ since $f(x) > 0$ for all $x \in X$.

Secondly suppose that $X \subseteq \mathbb{R}$ is not closed in $\mathbb{R}$, and let $c \in \overline{X} \setminus X$. Define $f : X \to \mathbb{R}$ by $f(x) = |x - c|$. Then the lower bound of $f$ is $0$ since for any

$\delta > 0$ there exists $x \in X$ with $|x - c| < \delta$. But this lower bound is not attained since $f(x) > 0$ for all $x \in X$.

13.11 This follows from Exercise 13.6 since the family $\{V_n : n \in \mathbb{N}\}$ has the finite intersection property - the intersection of any finite subfamily $\{V_{n_1}, V_{n_2}, \ldots, V_{n_r}\}$ is $V_N$ where $N = \max\{n_1, n_2, \ldots, n_r\}$ and $V_N \neq \emptyset$. Hence since $X$ is compact, Exercise 13.6 says that $\bigcap_{n=1}^{\infty} V_n$ is non-empty.

13.13 (a) We show that the $X_n$ form a nested sequence of closed subsets of $X$. Note that $X_1 = f(X_0) = f(X) \subseteq X_0$. Suppose inductively that $X_n \subseteq X_{n-1}$ for some integer $n \geqslant 1$. Then $f(X_n) \subseteq f(X_{n-1})$ which says $X_{n+1} \subseteq X_n$. So by induction $X_n \subseteq X_{n-1}$ for all $n \in \mathbb{N}$. Also, since $X$ is compact and $f$ is continuous, $X_1 = f(X)$ is compact by Proposition 13.15. Suppose that $X_n$ is compact for some integer $n \geqslant 0$. Then since $f$ is continuous, $X_{n+1} = f(X_n)$ is also compact by Proposition 13.15. By induction $X_n$ is compact for all integers $n \geqslant 0$. But $X$ is Hausdorff, so each $X_n$ is closed in $X$. Also, each $X_n$ is non-empty by inductive construction. Now by Exercise 13.11, $A = \bigcap_{n=0}^{\infty} X_n$ is non-empty.

(b) The inclusion $f(A) \subseteq A$ is straightforward: if $a \in A$ then $a \in X_n$ for any integer $n \geqslant 0$, so $f(a) \in f(X_n) = X_{n+1}$ for any integer $n \geqslant 0$. But since $X_0 \supseteq X_1 \supseteq \ldots \subseteq X_n \supseteq \ldots$, we have $\bigcap_{n=0}^{\infty} X_{n+1} = \bigcap_{n=1}^{\infty} X_n = \bigcap_{n=0}^{\infty} X_n = A$, and we see that $f(a) \in A$.

To prove the other inclusion we follow the hint: for any $a \in A$ let $V_n = f^{-1}(a) \cap X_n$. Since the $X_n$ are nested so are the $V_n$. Since $X$ is Hausdorff, $\{a\}$ is closed in $A$, hence since $f$ is continuous, $f^{-1}(a)$ is closed in $X$ by Proposition 9.5. Also each $X_n$ is closed in $X$ as in (a). So each $V_n$ is closed in $X$. Moreover, for each integer $n \geqslant 0$ we know $a \in X_{n+1} = f(X_n)$ so there exists $x \in X_n$ such that $f(x) = a$. This says that $V_n = f^{-1}(a) \cap X_n \neq \emptyset$. Now by Exercise 13.11, $\bigcap_{n=0}^{\infty} V_n$ is non-empty. Let $b$ be a point in this set. Then $b \in V_n = f^{-1}(a) \cap X_n$ for any integer $n \geqslant 0$. Also, $b = f^{-1}(a)$ says that $f(b) = a$, and $b \in X_n$ for all integers $n \geqslant 0$ says that $b \in \bigcap_{n=0}^{\infty} X_n = A$. So $A \subseteq f(A)$. We have now proved that $f(A) = A$.

13.15 The exercise gives the procedure for constructing the sequences $(a_n)$, $(b_n)$. The sequence $(a_n)$ converges to some real number $c$ since it is monotonic increasing and bounded above (for example by $b_1$ ). Likewise since $(b_n)$ is monotonic decreasing and bounded below (for example by $a_1$ ) it too converges to some real number $d$. Note that for any $m \geqslant n$ we have $a_n \leqslant a_m \leqslant b_m$, so in the limit as $m \to \infty$ we have $a_n \leqslant d$. This is true for all $n \in \mathbb{N}$ so in the limit as $n \to \infty$ we have $c \leqslant d$. For any $n \in \mathbb{N}$ we have $a_n \leqslant c \leqslant d \leqslant b_n$. Since $b_n - a_n = (b-a)/2^n$, also $d - c \leqslant (b-a)/2^n$. This gives $c = d$. But $c \in U$ for some $U \in \mathcal{U}$, and $U$ is open in $\mathbb{R}$, so there exists $\varepsilon > 0$ such that $(c - \varepsilon, c + \varepsilon) \subseteq U$. Hence for large enough $n$ the interval $[a_n, b_n] \subseteq U$ (since $c \in [a_n, b_n]$ and $b_n - a_n = (b-a)/2^n$ ). This contradicts the construction. So $\mathcal{U}$ must have a finite subcover after all.

13.17 Suppose that $\mathcal{T}_1,\ \mathcal{T}_2$ are topologies on a set $X$ such that $\mathcal{T}_1 \subseteq \mathcal{T}_2$, and that $(X,\,\mathcal{T}_1)$ is Hausdorff and $(X,\,\mathcal{T}_2)$ is compact. Consider the identity function from $(X,\,\mathcal{T}_2)$ to $(X,\,\mathcal{T}_1)$. This is continuous since $\mathcal{T}_1 \subseteq \mathcal{T}_2$, and it is one-one onto. Since $(X,\,\mathcal{T}_1)$ is Hausdorff and $(X,\,\mathcal{T}_2)$ is compact it follows from the inverse function theorem Proposition 13.26 that the identity function from $(X,\,\mathcal{T}_2)$ to $(X,\,\mathcal{T}_1)$ is a homeomorphism, i.e. the identity function from $(X,\,\mathcal{T}_1)$ to $(X,\,\mathcal{T}_2)$ is also continuous. This says that $\mathcal{T}_2 \subseteq \mathcal{T}_1$. So $\mathcal{T}_1 = \mathcal{T}_2$ as required.

In particular if $X = [0,\,1]$ and $\mathcal{T}_1$ is a Hausdorff topology on $[0,\,1]$ which is contained in the Euclidean topology $\mathcal{T}_2$, then since we know that $([0,\,1],\,\mathcal{T}_2)$ is compact, the first part of this exercise shows that $\mathcal{T}_1 = \mathcal{T}_2$. So $\mathcal{T}_1$ is not strictly coarser than the Euclidean topology $\mathcal{T}_2$.

13.19 Let $X$ be a compact Hausdorff space, let $W \subseteq X$ be closed in $X$ and let $y \in X \setminus W$. Since $W$ is a closed subspace of a compact space, $W$ is compact by Proposition 13.20. For any $w \in W$, by the Hausdorff condition there exist disjoint open subsets $U_w,\,V_w$ of $X$ such that $y \in U_w,\ w \in V_w$. Now $\{V_w : w \in W\}$ is an open cover of compact $W$ , so there is a finite subcover $\{V_{w_1},\,V_{w_2},\,\ldots,\,V_{w_r}\}$. Put

$$U = \bigcap_{i=1}^{r} U_{w_i}, \quad V = \bigcup_{i=1}^{r} V_{w_i}.$$

Then $U,\,V$ are open in $X$. Also, $W \subseteq V$ since $\{V_{w_1},\,V_{w_2},\,\ldots,\,V_{w_r}\}$ covers $W$. Next, $y \in U$ since $y \in U_{w_i}$ for all $i \in \{1,\,2,\,\ldots,\,r\}$. Finally $U$ and $V$ are disjoint, since for any point $v \in V$ we have $v \in V_{w_i}$ for some $i \in \{1,\,2,\,\ldots,\,r\}$ so since $U_{w_i}$ and $V_{w_i}$ are disjoint, $v \notin U_{w_i}$ hence $v \notin U$. This proves that $X$ is regular.

The proof that $X$ is normal is similar. Suppose that $W,\,Y$ are disjoint closed subsets of $X$. From the first part of this exercise, for each $y \in Y$ there exist disjoint open subsets $U_y,\,V_y$ of $X$ such that $y \in U_y,\ W \subseteq V_y$. Now $\{U_y : y \in Y\}$ is an open cover of $Y$, and $Y$ is compact by Proposition 13.20, so there is a finite subcover $\{U_{y_1},\,U_{y_2},\,\ldots,\,U_{y_s}\}$. Put

$$U = \bigcup_{j=1}^{s} U_{y_j}, \quad V = \bigcap_{j=1}^{s} V_{y_j}.$$

Then $U,\,V$ are open in $X$. Also, $Y \subseteq U$ since $\{U_{y_1},\,U_{y_2},\,\ldots,\,U_{y_s}\}$ is a cover for $Y$. Next, $W \subseteq V$ since $W \subseteq V_{y_j}$ for all $j \in \{1,\,2,\,\ldots,\,s\}$. Finally $U$ and $V$ are disjoint, since if $u \in U$ then $u \in U_{y_j}$ for some $j \in \{1,\,2,\,\ldots,\,s\}$ so $u \notin V_{y_j}$ since $U_{y_j}$ and $V_{y_j}$ are disjoint, so $u \notin V$. Hence $X$ is normal.

13.21 First, following the hint we prove that

$$f^{-1}(V) = p_X(G_f \cap p_Y^{-1}(V)) \text{ for any subset } V \subseteq Y\ .$$

For if $x \in f^{-1}(V)$ then $f(x) \in V$, and $x = p_X(x,\,f(x))$ where $(x,\,f(x)) \in G_f$ and also $(x,\,f(x)) \in p_Y^{-1}(V)$ since $p_Y(x,\,f(x)) = f(x) \in V$.

Suppose $x \in p_X(G_f \cap p_Y^{-1}(V))$. There exists $y \in Y$ such that $(x,\,y) \in G_f \cap p_Y^{1}(V)$. Now $(x,\,y) \in G_f$ says that $y = f(x)$ and $(x,\,y) \in p_Y^{-1}(V)$ says that $y \in V$, so $f(x) \in V$ so $x \in f^{-1}(V)$ as required.

Still following the hint we apply the above to a closed subset $V \subseteq Y$. The graph $G_f$ is given to be closed in $X \times Y$ and $p_Y^{-1}(V)$ is closed in $X \times Y$ by Proposition

9.5 since $V$ is closed in $Y$. So $G_f \cap p_Y^{-1}(V)$ is closed in $X \times Y$ and since $Y$ is compact, by Exercise 13.20 (a) $p_X(G_f \cap p_Y^{-1}(V))$ is closed in $X$. This tells us that $f^{-1}(V)$, which equals $p_X(G_f \cap p_Y^{-1}(V))$, is closed in $X$ whenever $V$ is closed in $Y$, so $f$ is continuous by Proposition 9.5.

# Chapter 14

14.1 Consider the sequence $(1/n)$ in $(0, 1)$. This has no subsequence converging to a point of $(0, 1)$ since the sequence $(1/n)$, and hence every subsequence of it, converges in $\mathbb{R}$ to $0$.

14.3 Let $A$ be a closed subset of a sequentially compact metric space $X$. Let $(x_n)$ be any sequence in $A$. Then $(x_n)$ is also a sequence in $X$, which is sequentially compact, hence there is a convergent subsequence $(x_{n_r})$. The point this converges to must lie in $A$ since $A$ is closed in $X$ (see Corollary 6.30). Hence $A$ too is sequentially compact.

14.5 Let $(y_n)$ be a sequence in $f(X)$. For each $n \in \mathbb{N}$ there is a point $x_n \in X$ such that $y_n = f(x_n)$. Since $X$ is sequentially compact there is some subsequence $(x_{n_r})$ of $(x_n)$ which converges to a point $x \in X$. Then by continuity of $f$ the subsequence $(y_{n_r}) = (f(x_{n_r}))$ converges in $Y$ to $f(x)$ (see Exercise 6.25). Hence $f(X)$ is sequentially compact.

14.7 This follows from Exercises 14.5 and 14.2. For if $f : X \to Y$ is a continuous map of metric spaces and $X$ is sequentially compact, then by Exercise 14.5 so is $f(X)$, and hence by Exercise 14.2 $f(X)$ is bounded.

14.9 Suppose that $(X, d_X)$ and $(Y, d_Y)$ are sequentially compact metric spaces. In $X \times Y$ we shall use the product metric $d_1$: recall that

$$d_1((x, y), (x', y')) = d_X(x, x') + d_Y(y, y').$$

Let $((x_n, y_n))$ be any sequence in $X \times Y$. First, since $X$ is sequentially compact there is a subsequence $(x_{n_r})$ of $(x_n)$ converging to a point $x \in X$. Now consider the sequence $(y_{n_r})$ in $Y$. Since $Y$ is sequentially compact there exists a subsequence $(y_{n_{r_s}})$ of $(y_{n_r})$ converging to a point $y$ in $Y$. Then $(x_{n_{r_s}})$ is a subsequence of $(x_{n_r})$ hence also converges to $x$. Consider the subsequence $((x_{n_{r_s}}, y_{n_{r_s}}))$ of $((x_n, y_n))$. This converges to $(x, y)$. For let $\varepsilon > 0$. Since $(x_{n_{r_s}})$ converges to $x$, there exists $S_1 \in \mathbb{N}$ such that $d_X(x_{n_{r_s}}, x) < \varepsilon/2$ whenever $s \geqslant S_1$. Similarly there exists $S_2 \in \mathbb{N}$ such that $d_Y(y_{n_{r_s}}, y) < \varepsilon/2$ whenever $s \geqslant S_2$. Put $S = \max\{S_1, S_2\}$. If $s \geqslant S$ then

$$d_1((x_{n_{r_s}}, y_{n_{r_s}}), (x, y)) = d_X(x_{n_{r_s}}, x) + d_Y(y_{n_{r_s}}, y) < \varepsilon.$$

So $((x_n, y_n))$ has a subsequence converging to a point in $X \times Y$. This shows that $X \times Y$ is sequentially compact. (As we have seen, any product metric gives the same answer.)

14.11 Let $x_n \in V_n$ for each $n \in \mathbb{N}$. Since $X$ is sequentially compact, there is a subsequence $(x_{n_r})$ of $(x_n)$ converging to some point in $X$. Since the $V_n$ are nested, $x_{n_r} \in V_m$ for all $r$ such that $n_r \geqslant m$. But $V_m$ is closed in $X$, so $x \in V_m$ (by Corollary 6.30). This holds for all $m \in \mathbb{N}$ so $x \in \bigcap_{n=1}^{\infty} V_n$ and this intersection is non-empty.

14.13 The exercise does most of this! As suggested, we prove inductively that $[a, a_i] \subseteq A$ for $a_1, a_2, \ldots, a_n = b$. This is true for $i = 1$ since $a_0 = a \in A$, and since $a_1 - a_0 < \varepsilon$ where $\varepsilon$ is a Lebesgue number for the cover $\{A, B\}$, we know that $[a_0, a_1]$ is contained in a single set of the cover, and this must be $A$ since $A \cap B = \emptyset$. Suppose inductively that $[a, a_i] \subseteq A$ for some $i \in \{1, 2, \ldots n-1\}$. We can repeat the above argument with $a$ replaced by $a_{n-1}$ and deduce that also $[a_{n-1}, a_n] \subseteq A$. Hence $[a, b] \subseteq A$, so $\{A, B\}$ is not a partition of $[a, b]$ after all. So $[a, b]$ is connected.

14.15 If, say, $V_{n_0}$ is empty, then $\bigcap_{n=1}^{\infty} V_n = \emptyset$ and its diameter is $0$ by definition. Likewise in this case diam $V_{n_0} = 0$ so

$$\inf\{\operatorname{diam} V_n : n \in \mathbb{N}\} = 0 \text{ also.}$$

Suppose now that all the $V_n$ are non-empty. (We already know from Exercise 14.11 that their intersection is then non-empty.) Now $\bigcap_{n=1}^{\infty} V_n \subseteq V_m$ for any $m \in \mathbb{N}$ so diam $\left(\bigcap_{n=1}^{\infty} V_n\right) \leqslant$ diam $V_m$. Hence

$$\operatorname{diam}\left(\bigcap_{n+1}^{\infty} V_n\right) \leqslant \inf\{\operatorname{diam} V_m : m \in \mathbb{N}\} = m_0 \text{ say.}$$

Conversely $m_0$ is a lower bound for the diameters of the $V_n$ so for any $\varepsilon > 0$ and any $n \in \mathbb{N}$ we know that diam $V_n > m_0 - \varepsilon$. Hence there exist points $x_n, y_n \in V_n$ such that $d(x_m, x_n) > m_0 - \varepsilon$. Since $X$ is sequentially compact $(x_n)$ has a subsequence $(x_{n_r})$ converging to a point $x \in X$, and then $(y_{n_r})$ has a subsequence $(y_{n_{r_s}})$ converging to a point $y \in X$. Since $(x_{n_{r_s}})$ is a subsequence of $(x_{n_r})$ it too converges to $x$. Also, by continuity of the metric, $d(x_{n_{r_s}}, y_{n_{r_s}}) \to d(x, y)$ as $s \to \infty$. Hence $d(x, y) \geqslant m_0 - \varepsilon$. Also, $x, y \in V_n$ for each $n \in \mathbb{N}$ since $V_n$ is closed in $X$. Since this is true for all $n \in \mathbb{N}$ we have $x, y \in \bigcap_{n=1}^{\infty} V_n$. Hence diam $\bigcap_{n=1}^{\infty} V_n \geqslant m_0 - \varepsilon$. But this is true for any $\varepsilon > 0$ , so diam $\bigcap_{n=1}^{\infty} V_n \geqslant m_0$.
The above taken together prove the result.

14.17 (a) Let $x \in X$. We want to show that $x \in f(X)$. Consider the sequence $(x_n)$ in $X$ defined by

$$x_1 = x, \quad x_{n+1} = f(x_n) \text{ for all integers } n \geqslant 1.$$

Since $X$ is sequentially compact there is a convergent subsequence, say $(x_{n_r})$. Any convergent sequence is Cauchy, so given $\varepsilon > 0$ there exists $R \in \mathbb{N}$ such that $|x_{n_r} - x_{n_s}| < \varepsilon$ whenever $s > r \geqslant R$. Now we use the isometry condition, iterated $n_R - 1$ times, to see that $|x_1 - x_{n_r - n_R + 1}| < \varepsilon$ whenever $r \geqslant R$. But $x_1 = x$ and $x_{n_r - n_R + 1} \in f(X)$ whenever $r > R$. Hence $x \in \overline{f(X)}$. But $X$ is compact and $f$ is continuous, so $f(X)$ is compact. Also, $X$ is metric so Hausdorff, hence $f(X)$ is closed in $X$. Hence $\overline{f(X)} = f(X)$. So $x \in f(X)$ for any $x \in X$ , which says that $f$ is onto. Hence $f$ is an isometry.

(b) We can apply (a) to the compositions $g \circ f : X \to X$ and $f \circ g : X \to X$ to see that these are both onto. Since $g \circ f$ is onto, $g$ is onto. Similarly since $f \circ g$ is onto, $f$ is onto. Hence both $f$ and $g$ are isometries.

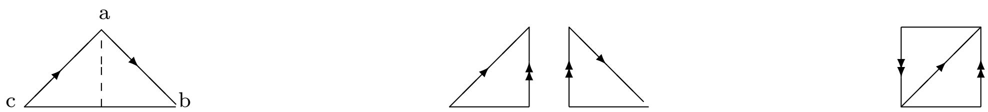
(c) We define $f : (0, \infty) \to (0, \infty)$ by $f(x) = x + 1$.

# Chapter 15

15.1 (a) A figure-of-eight.

(b) A two-dimensional sphere. (See below.)


(c) This gives a Möbius band. (See below.) For we first cut along the dashed vertical line in the triangle $abc$ to get two triangles as in the middle picture, with vertical sides labelled to recall how these are to be stuck together. Then we reassemble these middle triangles to get the picture on the right, which represents a Möbius band.




15.3 It is clear that $(0, t) \sim (1, 1-t)$ for any $t \in [0, 1]$, by (iv). What needs proving is that the equivalence classes are no bigger than the sets described in the question.

First if $0 < s < 1$ and $(s_2, t_2) \sim (s, t)$ then (i) must apply and $(s_2, t_2) = (s, t)$. So $(s, t)$ cannot be equivalent to any point other than itself - $\{(s, t)\}$ is a complete equivalence class.

Consider now what points can be equivalent to $(0, t)$, for some $t \in [0, 1]$. So suppose that $(s_2, t_2) \sim (0, t)$. Then *either* (i) applies and $(s_2, t_2) = (0, t)$ *or* (ii) applies and $s_2 = 1$ and $t_2 = 1 - t$. Thus the set $\{(0, t), (1, 1-t)\}$ is a complete equivalence class.

Thus each equivalence class is either a singleton set $\{(s, t)\}$ with $0 < s < 1$ and $t \in [0, 1]$ or a set containing two elements $\{(0, t), (1, 1-t)\}$ for some $t \in [0, 1]$.

15.5 For example since $[0, \pi) = (-\infty, \pi) \cap [0, 2\pi]$ it follows that $[0, \pi)$ is open in $[0, 2\pi]$. Now $f([0, \pi))$ is the upper semi-circle of $S^1$, closed at the end with co-ordinates $(1, 0)$ and open at the end $(-1, 0)$. This is not open in $S^1$, since any open set in $S^1$ containing the point $(1, 0)$ also contains points $(x, y) \in S^1$ with $y < 0$, and such points are not in $f([0, \pi))$.

15.7 This is proved by complements. By Proposition 3.9, $X \setminus f^{-1}(V) = f^{-1}(Y \setminus V)$ for any subset $V \subseteq Y$. Suppose that $f : X \to Y$ is a quotient map. Then $V \subseteq Y$ is closed in $Y$ iff $Y \setminus V$ is open in $Y$, which happens iff $X \setminus f^{-1}(V) = f^{-1}(Y \setminus V)$ is open in $X$ iff $f^{-1}(V)$ is closed in $X$.

Conversely suppose that $V \subseteq Y$ is closed in $Y$ iff $f^{-1}(V)$ is closed in $X$. Then $U \subseteq Y$ is open in $Y$ iff $Y \setminus U$ is closed in $Y$ iff $X \setminus f^{-1}(U) = f^{-1}(Y \setminus U)$ is closed in $X$ iff $f^{-1}(U)$ is open in $X$, so $f$ is a quotient map.

15.9 We proceed in the same way as in the proof that the quotient space of the square $S = [0, 2\pi] \times [0, 2\pi]$ by an appropriate equivalence relation is homeomorphic to $T$. Define

$$f : \mathbb{R}^2 \to T \text{ by } f((s, t)) = ((a + r\cos t)\cos s,\, (a + r\cos t)\sin s,\, r\sin t).$$

Then as in the proof mentioned above, we show that
(a) $f$ is a map to $T$,
(b) $i \circ f : \mathbb{R}^2 \to \mathbb{R}^3$ is continuous, where $i : T \to \mathbb{R}^3$ is the inclusion, so $f$ is continuous by Proposition 10.6.
(c) $f$ respects the equivalence relation $\sim$ just as before.

Hence $f$ induces a well-defined continuous map $g : \mathbb{R}^2/\sim \to T$. Just as before (with $S$ replacing $\mathbb{R}^2$ in the proof), we can show that $g$ is one-one onto $T$. Since $T$ is Hausdorff as a subspace of $\mathbb{R}^3$, it remains to prove that $\mathbb{R}^2/\sim$ is compact and this is really the only difference from the proof when $\mathbb{R}^2$ is replaced by $S$. But if $j : S \to \mathbb{R}^2$ is the inclusion map then the composition $p \circ j$ is onto where $p : \mathbb{R}^2 \to \mathbb{R}^2/\sim$ is the quotient map: for given any point $(s, t) \in \mathbb{R}^2$ we can find a point $(s', t') \in S$ such that $p((s', t')) = p((s, t))$. [If $S$ were the square $[0, 1] \times [0, 1]$ we could take $s' = s - [s]$ and $t' = t - [t]$ where $[s], [t]$ are the integer parts of $s, t$. Since $S$ is actually $[0, 2\pi] \times [0, 2\pi]$ we scale this and take $s' = 2\pi(s/2\pi - [s/2\pi])$ and $t' = 2\pi(t/2\pi - [t/2\pi])$. Hence since $S$ is compact so is $\mathbb{R}^2/\sim$ and as before we can apply Corollary 13.27 to see that $g$ is a homeomorphism.

# Chapter 16

16.1 Let $f$ be the pointwise limit of $(f_n)$ on $D$, which certainly exists since each $x \in D$ belongs to some $D_i$ , so $(f_n(x))$ converges. Let $\varepsilon > 0$. For each $i \in \{1, 2, \ldots, r\}$ the sequence $(f_n)$ converges uniformly on $D_i$, so there exists $N_i \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ whenever $n \geqslant N_i$ and $x \in D_i$. Let $N = \max\{N_1, N_2, \ldots, N_r\}$, let $x$ be any point in $D$ and let $n \geqslant N$. Then $x \in D_i$ for some $i \in \{1, 2, \ldots, r\}$ and $n \geqslant N \geqslant N_i$, so $|f_n(x) - f(x)| < \varepsilon$. Thus $(f_n)$ converges uniformly on $D$.

16.3 (i) Since $0 \leqslant x/(1+nx) \leqslant 1/n$ for all $x \in [0\ 1]$ ( including $x = 0$ ) it follows that the pontwise limit of $(f_n)$ here is the zero function. The same inequality shows that the maximum $M_n$ of $f_n$ on $[0, 1]$ satisfies $M_n \leqslant 1/n$ so $M_n \to 0$ as $n \to \infty$ and convergence is uniform on $[0, 1]$.

(ii) Again the pointwise limit exists and is the zero function: for $f_n(0) = 0$ for all $n \in \mathbb{N}$, and when $x \in (0, 1]$ we have $0 < f_n(x) < 2nx/(nx^2)^2 = 2/nx^3$. But $f_n'(x) = n\mathrm{e}^{-nx^2} - 2n^2x^2\mathrm{e}^{-nx^2}$ which is zero when $x = \sqrt{1/2n}$. Also, $f_n'(x) > 0$ for $x < \sqrt{1/2n}$ and $f_n'(x) < 0$ for $x = \sqrt{1/2n}$, so the maximum $M_n$ of $f_n$ on $[0, 1]$ is attained at $\sqrt{1/2n}$. Hence $M_n = \sqrt{n/2}\mathrm{e}^{-1/2}$. Since $M_n \not\to 0$ as $n \to \infty$ convergence is not uniform on $[0, 1]$.

(iii) Here $f_n(x) = 0$ for all $n \in \mathbb{N}$ when $x = 0$ or $1$ . For $x \in (0, 1)$ Exercise 4.9 shows that $f_n(x) \to 0$ as $n \to \infty$. So again $(f_n(x))$ converges pointwise to the zero function. Now $f_n'(x) = n^2x^{n-1}(1-x) - nx^n$, which is zero when $x = n/(n+1)$. Since $f_n(0) = f_n(1) = 0$ and $f_n(n/(n+1)) > 0$, we get

$$M_n = f_n\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^{n+1} = \left(\frac{1}{1+1/n}\right)^n \frac{n}{n+1} \to \frac{1}{\mathrm{e}} \neq 0$$

as $n \to \infty$. Hence convergence is not uniform on $[0, 1]$.

(iv) Again $f_n(0) = f_n(1) = 0$ for all $n \in \mathbb{N}$ and $f_n(x) \to 0$ as $n \to \infty$ for $x \in (0, 1)$ by Exercise 4.9, so again the pointwise limit is the zero function. Now $f_n'(x) = n^{1/2}(1-x)^n - n^{3/2}x(1-x)^{n-1}$. This is zero for $x = 1/(n+1)$. Since $f_n(1/(n+1)) > 0$ and $f_n(0) = f_n(1) = 0$ we have

$$M_n = f_n\left(\frac{1}{n+1}\right) = \frac{n^{1/2}}{n+1}\left(\frac{n}{n+1}\right)^n \to 0 \text{ as } n \to \infty.$$

So convergence is uniform on $[0, 1]$.

(v) Again as before $(f_n)$ converges pointwise to the zero function. The derivative $f_n'(x) = n(1-x^2)^{n^2} - 2x^2n^3(1-x^2)^{n^2-1}$ which is zero when $x = 1/\sqrt{2n^2+1}$. As before then

$$M_n = f_n(1/\sqrt{2n^2+1}) = \frac{n}{\sqrt{2n^2+1}}\left(\frac{2n^2}{2n^2+1}\right)^{n^2} \to \frac{1}{\sqrt{2e}}$$

as $n \to \infty$. Hence convergence is not uniform on $[0, 1]$ in this case.

(vi) Here the limit function is discontinuous - it takes the value $0$ when $x \in [0, 1)$ but the value $1/2$ when $x = 1$. Each $f_n$ is continuous on $[0, 1]$ so by Theorem 16.10 convergence is not uniform on $[0, 1]$.

(vii) The pointwise limit is again the zero function (we can argue separately for $x \in [0, 1)$ and for $x = 1$ ). The function $f_n$ satisfies $|f_n(x)| \leqslant x^n \leqslant 1/2^n$ for $x \in [0, 1/2]$, while $|f_n(x)| \leqslant 1/n^{1/2}$ for $x \in [1/2, 1]$. These give uniform convergence on $[0, 1/2]$ and $[1/2, 1]$ respectively hence Exercise 16.1 gives uniform convergence on $[0, 1]$.

16.5 Using 16.4 (b) let $K$ be a uniform bound for the set $\{f_n : n \in \mathbb{N}\}$, and using 16.4 (a) let $L$ be an upper bound for $|g(x)|$ on $D$. We may assume that $K, L$ are strictly positive. Let $\varepsilon > 0$. By uniform convergence of $(f_n)$ to $f$ , there exists $N_1 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon/2L$ for all $n \geqslant N_1$ and all $x \in D$. Similarly there exists $N_2 \in \mathbb{N}$ such that $|g_n(x) - g(x)| < \varepsilon/2K$ for all $n \geqslant N_2$

and all $x \in D$. Put $N = \max\{N_1, N_2\}$. Then for any $n \geqslant N$ and any $x \in D$ we have

$$\begin{aligned}|f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)(g_n(x) - g(x)) + g(x)(f_n(x) - f(x))| \\ &\leqslant |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| < K\varepsilon/2K + L\varepsilon/2L = \varepsilon.\end{aligned}$$

Hence $(f_n g_n)$ converges to $fg$ uniformly on $D$.

16.7 We may let

$$f_n(x) = \begin{cases} 1/n & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then $(f_n)$ converges uniformly on $\mathbb{R}$ although each $f_n$ is discontinuous at $0$ .

If we wish, we may even make each $f_n$ discontinuous everywhere by setting

$$f_n(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1/n & \text{if } x \notin \mathbb{Q}. \end{cases}$$

16.9 We first make a reduction of the problem by setting $g_n(x) = f_n(x) - f(x)$, so that $(g_n)$ converges pointwise to the zero function and we still have that each $g_n$ is continuous on $X$ and that $g_n(x) \geqslant g_{n+1}(x)$ for all $n \in N$ and all $x \in X$.

Let $\varepsilon > 0$. By pointwise convergence, for any $x \in X$ there exists an integer $N_x$ such that $0 \leqslant g_n(x) < \varepsilon/2$ for all $n \geqslant N_x$. Now $g_{N_x}$ is continuous at $x$ so there exists an open set $U_x \subseteq X$ containing $x$ and such that $|g_{N_x}(y) - g_{N_x}(x)| < \varepsilon/2$ whenever $y \in U_x$. Hence $0 \leqslant g_{N_x}(y) < \varepsilon$ whenever $y \in U_x$. Hence by monotonicity $0 \leqslant g_n(y) < \varepsilon$ whenever $y \in U_x$ and $n \geqslant N_x$. Now $\{U_x : x \in X\}$ is an open cover of the compact space $X$ , so there exists a finite subcover, say $\{U_{x_1}, U_{x_2}, \ldots, U_{x_r}\}$. Let

$$N = \max\{N_{x_1}, N_{x_2}, \ldots, N_{x_r}\}.$$

Then for any $n \geqslant N$ and any $y \in X$ we have $y \in U_{x_i}$ for some $i \in \{1, 2, \ldots, r\}$. So $n \geqslant N \geqslant N_{x_i}$, and we get $0 \leqslant g_n(y) < \varepsilon$. So $(g_n)$ converges to the zero function uniformly on $X$ , hence $(f_n)$ converges to $f$ uniformly on $X$.

# Chapter 17

17.1 Let $(x_n)$ be a Cauchy sequence in a discrete metric space $(X, d)$. There exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < 1$ whenever $m \geqslant n \geqslant N$, so in fact $d(x_m, x_n) = 0$ whenever $m \geqslant n \geqslant N$ since the metric is discrete. This says that the sequence is 'eventually constant', i.e. $x_n = x_N$ for all $n \geqslant N$. So $(x_n)$ converges, hence $(X, d)$ is complete.

17.3 (a) Suppose that $A$ and $B$ are complete subspaces of a metric space $X$ and let $(x_n)$ be a Cauchy sequence in $A \cup B$. Then either $(x_n)$ has a subsequence which lies entirely within $A$, or it has a subsequence in $B$, or both of these are true. Such a subsequence is Cauchy since $(x_n)$ is, so it converges by completeness of $A$ (or of $B$ ). But if a Cauchy sequence has a convergent subsequence, the whole sequence converges by Lemma 17.10. Hence $A \cup B$ is complete.

(b) Suppose for each $i$ in some index set $I$ that $A_i$ is a complete subspace of a metric space $X$, and let $(x_n)$ be a Cauchy sequence in $\bigcap_{i\in I} A_i$ assuming this intersection is non-empty. Then for any particular $i_0 \in I$, the sequence $(x_n)$ is a Cauchy sequence in $A_{i_0}$ hence it converges. Hence $\bigcap_{i\in I} A_i$ is complete. (If this intersection is empty then it is vacuously complete since there are no Cauchy sequences in it for which to check convergence.)

17.5 Let us label these metrics $d_a$, $d_b$, $d_c$.

(a) Then $(\mathbb{R}, d_a)$ is complete. For let $(x_n)$ be a $d_a$-Cauchy sequence. Then $(x_n^3)$ is a Cauchy sequence in the ordinary sense, so it converges say to $y \in \mathbb{R}$. Let $x = y^{1/3}$. Then $(x_n)$ converges to $x$ in the metric $d_a$. This holds since $d_a(x_n, x) = |x_n^3 - x^3| = |x_n^3 - y| \to 0$ as $n \to \infty$.

(b) This is not complete. Consider the sequence $(-n)$. Then $\mathrm{e}^{-n} \to 0$ as $n \to \infty$, so $(\mathrm{e}^{-n})$ is Cauchy in the usual sense, so $(-n)$ is $d_b$-Cauchy. But suppose that $(-n)$ converged to a real number $x_0$ in $(\mathbb{R}, d_b)$. Then $d_b(-n, x_0) \to 0$ as $n \to \infty$, which says that $(\mathrm{e}^{-n})$ converges to $\mathrm{e}^{x_0}$ in the usual metric on $\mathbb{R}$. But we know that $(\mathrm{e}^{-n})$ converges to $0$ in the usual metric, so we'd have $\mathrm{e}^{x_0} = 0$. But there is no such real number $x_0$ , so $(-n)$ cannot converge in $(\mathbb{R}, d_b)$. Hence $(\mathbb{R}, d_b)$ is not complete.

(c) For similar reasons this is not complete either. Consider the sequence $(n)$ in $\mathbb{R}$. The sequence $(\tan^{-1}(n))$ converges to $\pi/2$ in the usual sense as $n \to \infty$. So $(\tan^{-1}(n))$ is Cauchy in the usual sense which says that $(n)$ is $d_c$-Cauchy. But suppose that $(n)$ converges to some $x_0 \in \mathbb{R}$ in the metric space $(\mathbb{R}, d_c)$. Then $(\tan^{-1}(n))$ converges to $\tan^{-1}(x_0)$ in the usual sense. This gives $\tan^{-1}(x_0) = \pi/2$. But there is no such real number $x_0$. So $(n)$ does not converge in $(\mathbb{R}, d_c)$ and this space is not complete.

17.7 (a) Suppose that $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leqslant K|x-y|^\alpha$ for some constants $K, \alpha$ with $\alpha > 0$ and all $x, y \in D$. Let $\varepsilon > 0$ and put $\delta = (\varepsilon/K)^{1/\alpha}$. Then whenever $x, y \in D$ and $|x-y| < \delta$ we have

$$|f(x) - f(y)| \leqslant K|x-y|^\alpha < K(\varepsilon/K)^{(1/\alpha)\alpha} = \varepsilon.$$

So $f$ is uniformly continuous on $D$.

(b) Suppose that for any $x, y \in [a, b]$ we have $|f(x) - f(y)| \leqslant K|x-y|^\alpha$ for some constants $K, \alpha$ with $\alpha > 1$. Then for any $x \in [a, b]$ and $h \in \mathbb{R}$ with $h > 0$ and $x + h \in [a, b]$ we get

$$0 \leqslant \frac{|f(x+h) - f(x)|}{|h|} \leqslant K|h|^{\alpha-1}, \text{ and as } h \to 0 \text{ we have}$$

$|h|^{\alpha-1} \to 0$ since $\alpha > 1$. This shows that $f$ is differentiable on $[a, b]$ with derivative zero everywhere in $[a, b]$, so $f$ is constant on $[a, b]$ (by the mean value theorem).

(c) By the mean value theorem, for any $x, y \in [a, b]$, $f(x) - f(y) = (x-y)f'(\xi)$ for some $\xi$ between $x$ and $y$ . The conclusion follows.

17.9 The derivative of the cosine function is the negative sine function, which satisfies

$0 \leqslant \sin x \leqslant \sin 1$ for $x \in [0, 1]$. So $\cos x$ is decreasing on $[0, 1]$, with $\cos 0 = 1$ and $\cos 1 > 0$. So the cosine function does map $[0, 1]$ into itself. Moreover, since $|\sin x| \leqslant \sin 1 < 1$ for $x \in [0, 1]$, by Exercise 17.7 (c) the cosine function is a contraction on $[0, 1$ and by Theorem 17.22 we can get successive approximations $x_n$ to a solution of $\cos x = x$ by iterating the cosine function beginning with $x_1 = 0.5$ say. This gives $x_2 = \cos x_1 = 0.88$, $x_3 = \cos x_2 = 0.64$, and successive $x_n$ thereafter as 0.80, 0.70, 0.76, 0.72, 0.75, 0.73, 0.745, 0.735, 0.737. So it begins to look as if the unique solution of $\cos x = x$ in $[0, 1]$ is 0.74 correct to two decimal places. To check this we calculate $\cos x - x$ at $x = 0.735$ and at $x = 0.745$, and get respectively positive and negative answers, so by the intermediate value theorem there must be a point between 0.735 and 0.745 where $\cos x - x = 0$, This checks that 0.74 is a solution of $\cos x = x$ correct to two decimal places. (There are faster ways to get an approximate solution. For example the Newton-Raphson method with the same starting point is correct to two decimal places after only two stages.)

17.11 Clearly $f$ has no fixed point in $(0, 1/4)$ since the only solutions of $x^2 = x$ are $x = 0, 1$. Now $f$ does map $(0, 1/4)$ into itself, since $0 < x^2 < 1/16$ when $0 < x < 1/4$. Moreover, for any $x, y \in (0, 1/4)$ we have

$$|f(x) - f(y)| = |x^2 - y^2| = |(x-y)(x+y)| \leqslant |x-y|/2,$$

so $f$ is a contraction of $(0, 1/4)$. But $(0, 1/4)$ is not complete, so this does not contradict Banach's fixed point theorem.

17.13 (a) By the contraction map theorem applied to $f^{(k)}$ there is a unique fixed point $p \in X$ , with $f^{(k)}(p) = p$. Now we have $f^{(k)}(f(p)) = f(f^{(k)})(p) = f(p)$, so $f(p)$ is a fixed point of $f^{(k)}$ and by uniqueness $f(p) = p$. This says that $p$ is a fixed point of $f$ itself. Moreover, if also $f(q) = q$ then by an easy induction we see that $f^{(n)}(q) = q$ for any $n \in \mathbb{N}$. In particular $f^{(k)}(q) = q$, so by uniqueness $q = p$. Hence $f$ itself has a unique fixed point in $X$.

Now for any fixed $x \in X$ consider the sequence $(f^{n)}(x))$. We wish to show that it converges to $p$. This is true for the subsequence $(f^{(nk)}(x))$ by the contraction map theorem, since $f^{(k)}$ is a contraction of $X$. Let $K$ be a contraction constant for $f^{(k)}$. Let $\varepsilon > 0$. Since $(f^{(nk)})$ converges to $p$ there exists $N_1 \in \mathbb{N}$ such that $d(f^{(nk)}(x), p) < \varepsilon/2$ for all $n \geqslant N_1$, where $d$ is the metric on $X$. Let

$$M = \max\{d(f(x), p),\, d(f^{(2)}(x), p),\, \ldots,\, d(f^{(k-1)})(x), p)\}$$

and let $N_2 \in \mathbb{N}$ be such that $K^{N_2}M < \varepsilon/2$. (We may choose such an $N_2$ since $0 \leqslant K < 1$. ) Put $N = \max\{N_1, N_2\}$. Whenever $n \geqslant Nk$ we have $n = mk + i$ for some $m \geqslant M$ and some integer $i$ satisfying $0 \leqslant i < k$. We get

$$\begin{aligned} d(f^{(n)}(x), p) &\leqslant d(f^{(mk+i)}(x), f^{(mk)}(x)) + d(f^{(mk)}(x), p) \\ &\leqslant K^m d((f^{(i)}(x), x) + \varepsilon/2 \leqslant K^m M + \varepsilon/2 \leqslant K^{N_2}M + \varepsilon/2 < \varepsilon. \end{aligned}$$

This shows that $(f^{(n)}(x))$ converges to $p$.

(b) The idea for seeing that $\cos : \mathbb{R} \to \mathbb{R}$ is not a contraction is that its derivative can get arbitrarily close to 1. Suppose for a contradiction that $\cos : \mathbb{R} \to \mathbb{R}$ is a contraction with contraction constant $K < 1$. Since $\sin x \to 1$ as $x \to \pi/2-$ we may choose $\delta > 0$ such that $\sin x > K$ for all $x \in [\pi/2 - \delta, \pi/2]$. By the mean value theorem, $|\cos\pi/2 - \cos(\pi/2-\delta)| = |\delta\sin\xi|$ for some $\xi \in (\pi/2-\delta,\, \pi/2)$, so $|\cos\pi/2 - \cos(\pi/2-\delta)| > K\delta$, contradicting the contraction condition.

To see that $\cos^{(2)} : \mathbb{R} \to \mathbb{R}$ is a contraction, put

$$g(x) = \cos(\cos x). \text{ Then } g'(x) = \sin x \sin(\cos x),$$

and for any $x \in \mathbb{R}$ we have

$$|g'(x)| \leqslant |\sin(\cos x)| \leqslant \sin 1, 1.$$

so $g$ is a contraction by Exercise 17.7 (c).

17.15 Consider the composition

$$X \overset{\Delta}{\to} X \times X \overset{1 \times f}{\longrightarrow} X \times X \overset{d}{\to} \mathbb{R}.$$

Each map here is continuous by Propositions 5.19, 5.22 and Exercise 5.17 so the composition, $F$ say, is continuous by Proposition 5.18. Since $X$ is compact, $F$ attains its lower bound $l$, say, on $X$. Suppose that $l > 0$ and that it is attained at $x_0$, so $d(x_0,\, f(x_0)) = l$. Then $x_0$ and $f(x_0)$ are distinct, so by assumption $l = F((f(x_0)) = d(f(x_0),\, f(f(x_0))) < d(x_0,\, f(x_0)) = l$. This contradiction shows that $l$ must be $0$. Since $l$ is attained $d(p,\, f(p)) = 0$ for some $p \in X$, so $f(p) = p$ and we have proved the existence of a fixed point.

Uniqueness: for distinct fixed points $p,\, q$ we would have
$d(p,\, q) = f(f(p),\, f(q)) < d(p,\, q)$, a contradiction. So $p$ is the unique fixed point.