Math 145A
Winter 2016

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Solutions to practice problems

First note that **problem 7 in the list is valid as stated.** Also, in the first problem B should be a subset of X and not Y.

1. By definition a point x lies in $f^{-1}\big[\,[f[B]\big]$ if and only if $f(x) \in f[B]$, or equivalently $f(x) = f(b)$ for some $b \in B$. But $f(b) = f(b)$, and therefore $b \in f^{-1}\big[\,[f[B]\big]$.

To give an example where containment is proper, let $f(x) = x^2$ and let $B = [0,1]$. In this case $f^{-1}\big[\,[f[B]\big] = [-1,1]$.■

2. Suppose that $|x-1| < \delta$, where $0 < \delta < 1$. The first step is to see what that implies about $|f(x) - 1| = |f(x) - f(1)|$.

The starting point is the assumption that $1-\delta < x < 1+\delta$. This translates into the inequalities

$$\frac{1}{1+\delta} - 1 \quad < \quad \frac{1}{x} - 1 \quad < \quad \frac{1}{1-\delta} - 1$$

which simplifies to

$$\frac{-\delta}{1+\delta} \quad < \quad \frac{1}{x} - 1 \quad < \quad \frac{\delta}{1-\delta}\ .$$

Note that the expression on the right has a greater absolute value than the expression on the left. Since $f(x)$ is a strictly decreasing function, it is enough to find a value of δ such that

$$\frac{\delta}{1-\delta} \quad \leq \quad \varepsilon\ .$$

One easy way to do this is to solve for δ in terms of ε when the inequality is replaced by an equation. This can be done by standard algebra, and the result is $\delta = \varepsilon/(1+\varepsilon)$.■

3. Since $f(c) \geq f(x)$ for $x \leq c$, it follows that $f(c)$ is an upper bound for all such values, and hence $f(c)$ is greater than or equal to the greatest lower bound, which by definition is $f(c+)$. Similarly, since $f(c) \leq f(x)$ for $x \geq c$, it follows that $f(c)$ is a lower bound for all such values, and hence $f(c)$ is less than or equal to the least upper bound, which by definition is $f(c-)$.

Suppose now that f is continuous at c, and let $\varepsilon > 0$. If f is constant for $x \leq c$, then the conclusion is trivial because $x < c$ implies $f(x) = f(c)$, so assume it is not constant for the given values of x. Choose $c_1 < c$ such that $f(x_1) < f(c)$, and let ε' be the smaller of $f(c) - f(c_1)$ and ε. Then by the Intermediate Value Property of continuous real valued functions there is some c_2 between c_1 and c such that $f(c_2) > f(c) - \frac{1}{2}\varepsilon' \geq f(c) - \frac{1}{2}\varepsilon$. This means that $f(c)$ is a least upper bound for the set of all $f(x)$ such that $x < c$, and therefore $f(c) = f(c-)$.

Similarly, if f is continuous at c and $\varepsilon > 0$, the case where f is constant for $x \geq c$ is trivial, so assume that f is not constant for the give values of x. Then we can choose $b_1 > c$ such that $f(b_1) > f(c)$ and take ε' to be the smaller of $f(b_1) - f(c)$ and ε. As before there is some b_2 between b_1 and b such that $f(b_2) < f(c) + \frac{1}{2}\varepsilon' \leq f(c) - \frac{1}{2}\varepsilon$. This means that $f(c)$ is a greatest lower bound for the set of all $f(x)$ such that $x > c$, and therefore $f(c) = f(c+)$. This proves one direction of the implication in the exercise.

Conversely, suppose that $f(c-) = f(c) = f(c+)$, and let $\varepsilon > 0$. Then one can find $d_1 < c < d_2$ such that

$$f(c) - \varepsilon < f(d_1) \leq f(c) \leq f(d_2) < f(c) + \varepsilon$$

and hence $x \in (d_1, d_2)$ implies $|f(x) - f(c)| < \varepsilon$ **because f is an increasing function of x**. If we now take δ to be the smaller of $c - d_1$ and $d_2 - c_1$, then $(c - \delta, c + \delta) \subset (d_1, d_2)$, and therefore $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$.■

4. (i) Take $A = (0, 1)$ and $B = [1, 2)$.■
(ii) Take $A = [0, 2]$ and $B = [1, 3)$.■
(iii) Take $A = (0, 2]$ and $B = [1, 3)$.■
(iv) Take $A = [0, 2)$ and $B = (1, 3]$.■

5. Suppose that $b \notin A$; then we need to show that b is a limit point of A. Let $\varepsilon > 0$. Since b is the least upper bound, it follows that there is some $a \in A$ such that $a > b - \varepsilon$. This means that $a \in (N_\varepsilon(b) - \{b\}) \cap A$, and hence b is a limit point of A.■

6. (i) If A is open then $A = X \cap A$ where X is closed and A is open. If A is closed then $A = X \cap A$ where X is open and A is closed.■

(ii) *Disregard the remark in parentheses.* The set $[0, 1)$ is the intersection of the open subset $(-\infty, 1)$ and the closed set $[0, infty)$.■

(iii) Follow the hint, and assume that $\mathbb{Q} = E \cap V$ where E is closed and V is open. The intersection identity implies that E is a closed subset containing $\mathbb{Q}$. Since the closure of the latter is the real line, it follows that $\mathbb{Q} = V$ where V is open in $\mathbb{R}$. However, this is impossible because $\mathbb{Q}$ contains no open subset; specifically if $\varepsilon > 0$ and $1/n < \varepsilon$, then $q + \sqrt{2}/2n$ is an irrational number which lies in $(q - \varepsilon, q + \varepsilon)$.■

7. This is an exercise in using the definitions and manipulating inverse images. Recall that taking inverse images preserves unions, intersections and complements. Therefore we have

$$f^{-1}(A + B) = f^{-1}\Big[\big(A \cap (X - B)\big) \cup \big(B \cap (X - A)\big)\Big] =$$

$$f^{-1}\big[A \cap (X - B)\big] \cup f^{-1}\big[B \cap (X - A)\big]$$

$$\big(f^{-1}[A] \cap (Y - f^{-1}[B]\big) \cup \big(f^{-1}[B] \cap (Y - f^{-1}[A]\big) = f^{-1}[A] + f^{-1}[B]$$

which is (stronger than) what we wanted to prove.■

8. (i) Follow the hint; since the square root function is increasing on nonnegative real numbers, it is enough to show that the square of the left hand side is less than or equal to the square of the right hand side. This is true because

$$\big(\sqrt{u + v}\big)^2 = u + v \leq u + v = 2\sqrt{uv} = \big(\sqrt{u} + \sqrt{v}\big)^2 .■$$

(ii) Since $d \geq 0$ it follows that $\sqrt{d} \geq 0$, and of course we also have $\sqrt{d(x, x)} = \sqrt{0} = 0$ for all x. Conversely if $0 = \sqrt{d(x, y)}$ then $0 = d(x, y)$ and hence $x = y$. The symmetry proprerty follows because $d(x, y) = d(y, x)$ implies $\sqrt{d(x, y)} = \sqrt{d(y, x)}$. Finally, the Triangle Inequality follows because $\sqrt{d(x, y)} \leq \sqrt{d(x, z) + d(z, y)}$ because the square root function is increasing and

$$\sqrt{d(x, z) + d(z, y)} \leq \sqrt{d(x, z)} + \sqrt{d(z, y)}$$

by the first part of the exercise.■

(iii) This follows by systematically replacing $\sqrt{d}$ with $\varphi \circ d$ in the preceding argument. Note that for the function $\varphi(t) = Ct$ we have $\varphi(u + v) = \varphi(u) + \varphi(v)$.■