Math 145A
Winter 2016

UPDATED GENERAL INFORMATION — FEBRUARY 11, 2016

Formulas for inverse homeomorphisms

At the bottom of page 8.4 of `math145Anotes08.pdf` we gave specific examples of homeomorphisms from one closed interval $[a, b]$ to another closed interval $[c, d]$ and from $(-1, 1)$ to $\mathbb{R}$. For each of these functions f we claimed that the continuity of the inverse function could be shown by explicit calculation of the inverse function using the standard identity

$$x \quad = \quad f^{-1}(y) \qquad \Leftrightarrow \qquad y \quad = \quad f(x)$$

and for the linear case this is a straightforward exercise in algebra. For the function $f(x) = \tan\left(\frac{\pi}{2}\,x\right)$ this followed from the standard definition $\arctan y$, in which the values of the latter range over the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, but for the function

$$f(x) \quad = \quad \frac{x}{1 - |x|}$$

things are slightly more complicated and require a closer study of the function. In particular, one needs to examine two cases, one where $x \geq 0$ and one where $x \leq 0$. Fortunately, this is not difficult because $f(x) \geq 0$ if $x \geq 0$ and $f(x) \leq 0$ for $x \leq 0$.

If $x \geq 0$ then $y = f(x) = \frac{1}{1-x}$ and straightforward algebra shows that $x = f^{-1}(y) = \frac{y}{1+y}$. If $x \leq 0$ then $y = f(x) = \frac{1}{1+x}$ and straightforward algebra shows that $x = f^{-1}(y) = \frac{y}{1-y}$. Since x and y are either both nonnegative or both nonpositive, it turns out that

$$f^{-1}(y) \quad = \quad \frac{y}{1 + |y|}$$

in both cases, and hence there is a reasonably concise formula for the inverse function in this case too. Standard results on continuity imply that this inverse function is also continuous.

WARNING. There are many examples of homeomorphisms from $\mathbb{R}$ to itself which are given by reasonable formulas from single variable calculus but have inverses which cannot be expressed in terms of functions from single variable calculus. One specific example is the function $f(x) = x + e^x$. This function is 1–1 onto because it is strictly increasing (the derivative is positive everywhere) and hence is 1–1, and it is onto by the identities

$$-\infty \quad = \quad \lim_{x \to -\infty} \; f(x)\;, \qquad \infty \quad = \quad \lim_{x \to \infty} \; f(x)$$

plus the Intermediate Value Property. However, the inverse function is not given in terms of the "elementary functions" in calculus. Further information on this topic appears in the course directory file `lambert-fcn.pdf`.

Information regarding the second quiz

The second quiz will cover the material in Chapter 6 from the cutoff for the first midterm, and continuing through Chapters 7, 8 and 9. The quiz will contain questions about definitions and examples of basic concepts, taken from the following list:

1. Define the concept of limit point for a subset of a topological or metric space.

2. Define what it means for a function f from one topological space $(X_1, \mathcal{T}_1)$ to another space $(X_2, \mathcal{T}_2)$ to be a homeomorphism.

3. Define the indiscrete topology on a set X.

4. Define the cofinite topology on a set X.

5. Define the concept of a base for a topological space $(X, \mathcal{T})$.

6. Let $\mathcal{A}$ be a family of subsets for some set X, and let $\mathcal{T}$ denote the topology on X generated by $\mathcal{A}$. Describe the elements of $\mathcal{T}$ in terms of unions and intersections of sets in $\mathcal{A}$.

7. Give an example of an open subset U in a topological space X such that U is properly contained in $\text{Int}\,\overline{U}$. [*Hint:* There are simple examples where X is the real numbers with the usual topology.]

> *Note.* For every open subset $U \subset X$ we have $U \subset \text{Int}\,\overline{U}$, and for many standard examples the two subsets are equal.

8. Give an example of a closed subset F in a topological space X such that F properly contains $\overline{\text{Int}\,F}$. [*Hint:* There are simple examples where X is the real numbers with the usual topology.]

> *Note.* For every closed subset $F \subset X$ we have $F \supset \overline{\text{Int}\,F}$, and for many standard examples the two subsets are equal.

0. Give examples of subsets $A, B \subset \mathbb{R}$ such that the intersection of the closures $\overline{A} \cap \overline{B}$ properly contains $\overline{A \cap B}$.

> *Note.* For every pair of subsets $A, B \subset X$ we know that $\overline{A} \cap \overline{B}$ contains $\overline{A \cap B}$, and in some cases equality holds.

10. Give examples of subsets $A, B \subset \mathbb{R}$ such that the union of the interiors $\text{Int}\,A \cup \text{Int}\,B$ is properly contained in $\text{Int}\,A \cup B$.

> *Note.* For every pair of subsets $A, B \subset X$ we know that $\text{Int}\,A \cup \text{Int}\,B$ is contained in $\text{Int}\,A \cup B$, and in some cases equality holds.

11. Give an example of an infinite sequence of open subsets $U_n \subset \mathbb{R}$ such that the infinite intersection $\cap_n U_n$ is not open.

12. Give an example of a pair of subsets $A, B \subset \mathbb{R}$ such that A is closed, B is open but not closed, and $A \cup B$ is closed.

Answers for the last six questions are given in the file `quiz2review.pdf`. For each of these questions there is more than one correct answer.