Math 145A
Spring 2016

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Hints for the Examination 2 study problems

(X0) This set is the union of A and B, which are defined by $x \geq 0$ and $y \geq 0$. Each of these is connected because we can describe each subset as a product of connected spaces:

$$A \quad = \quad [0, \infty) \times \mathbb{R} \,, \qquad B \quad = \quad \mathbb{R} \times [0, \infty)$$

Since the intersection of the two subsets is the first quadrant where $x, y \geq 0$, this intersection is nonempty and there for $A \cup B$ must be connected.■

(X1) These are the open interval itself, the closed interval $[0, 1]$ and the half-open intervals $[0, 1)$ and $(0, 1]$.■

(X2) If $|x| + |y| \leq 1$, then $|x|, |y| \leq 1$ and therefore the distance between the points $(0, 0)$ and $|x|, |y|$ is at most

$$\sqrt{x^2 + y^2} \quad \leq \quad \sqrt{1 + 1 = 2}$$

which means that the set in question is bounded. It is also closed because it is the set of points where the value of the continuous function $|x| + |y|$ is ≤ 1, and a set of this form is closed. Since a subset of $\mathbb{R}^2$ is compact if and only if it is closed and bounded, the set in question must be compact.■

(X3) A compact set is bounded, and if we sketch the region in question it does not look bounded. Consider the points $(n, 1/2n)$, where n runs through all positive integers. These points all lie in the given set, and the n^{th} point a_n satisfies $d(a_n, 0) \geq n$. Therefore the given set is not bounded and hence cannot be compact.■

(X4) If we graph the function $y = f(x)$, where $f(x)$ is the rational expression in this question, we see that for $x \geq 0$ this graph is always above the line $y = \frac{3}{5}$ and this line is an asymptote for the graph. This should strongly suggest that $\frac{3}{5}$ is the greatest lower bound. More formally, we have

$$\lim_{x \to +\infty} \frac{3x + 4}{5x + 6} \quad = \quad \frac{3}{5}$$

which means that the values of the rational expression are with ε of the limit value for x sufficiently large, and we also have

$$\frac{3x + 4}{5x + 6} \quad > \quad \frac{3}{5}$$

because for positive numbers we have

$$\frac{a}{b} \quad > \quad \frac{c}{d}$$

if and only if $ad > bc$ and for our example we have

$$5(3x + 4) \quad = \quad 15x + 20 \quad > \quad 15x + 18 \quad = \quad 3(5x + 6)$$

when $x \geq 0$. Therefore we know that $\frac{3}{5}$ is a lower bound and for each $\varepsilon > 0$ we can find x such that $f(x) < \frac{3}{5} + \varepsilon$. — The relevance of this to Chapter 13 is that f is a continuous function on the nonnegative reals which is bounded from below but does not attain a minimum value.■

(X5) We can rewrite the defining equation for A in the form $y = 1$ or $y = -1$. Geometrically, the set consists of a pair of parallel horizontal lines, and this does not look like a connected set. To prove it actually is not connected, note that each of the lines $A_{\pm}$ defined by $y = \pm 1$ is closed in the plane, and hence in A. These lines are disjoint, and their union is A, and therefore we have a presentation of A as a union of two nonempty disjoint proper closed subsets, which means that A is not connected.■

(X6) Let Δ_A and Δ_B denote the diameters of A and B respectively, and pick points $a_0 \in A$ and $b_0 \in B$.

We need to show that if $x, y \in A \cup B$, then there is some $K > 0$ such that $d(x, y) \leq K$.

First case. Both points lie in A. Then the distance between then is bounded from above by the diameter of A.

Second case. Both points lie in B. Then the distance between then is bounded from above by the diameter of B.

Third case. One point lies in A and the other lies in B. We might as well label the points so that $x \in A$ and $y \in B$. In this case we apply the Triangle Inequality a couple of times:

$$d(x, y) \quad \leq \quad d(x, a_0) \; + \; d(a_0, b_0) \; + \; d(b_0, , y) \quad \leq \quad \Delta_A \; + \; d(a_0, b_0) \; + \; \Delta_B$$

In each of the three cases, the expression on the right of the display is an upper bound for $d(x, y)$.

Suppose now that $A \cap B$ is nonempty, and let $z \in A \cap B$. As before, let $x, y \in A \cup B$ and split the discussion into three cases as in the preceding portion of the argument. Once again we have $d(x, y) \leq \Delta_A$ or $d(x, y) \leq \Delta_B$ if $x, y \in A$ or $x, y \in B$. If (say) $x \in A$ and $y \in B$, then we have

$$d(x, y) \quad \leq \quad d(x, z) \; + \; d(y, z) \quad \leq \quad \Delta_A \; + \; \Delta_B$$

and therefore $\Delta_A + \Delta_B$ is an upper bound for $d(x, y)$ in all cases. — Although the problem does not ask for this, it is not difficult to find examples where the diameter of $A \cup B$ equals the sum of the diameters for A and B. One possibility is to let A and B be the closed intervals $[-1, 0]$ and $[0, 1]$.

For the final part, the idea is that one can have a pair of bounded subsets which are separated by an arbitrarily large distance. In particular, if we take the open intervals $[-1, 0]$ and $[n, n+1]$ we see that the diameter is $n + 2$ and hence can be made as large as we please.■

(X7) The image $f[X] \subset Y$ is compact because a continuous image of a compact set is always compact.

To prove the assertion about a maximum distance, consider the function $h(x) = d(f(x), y_0)$. This is continuous because it is a composite of two continuous functions; namely, the function $\varphi : X \to Y \times Y$ given by $\varphi(x) = (f(x), y_0)$ (which is continuous because its coordinates are) and the distance function $d : Y \times Y \to \mathbb{R}$.

One way to finish the argument is to note that a continuous real valued function on a compact set has a maximum value by the same sort of argument proving the maximum value property for continuous functions on closed intervals. However, we can also give a more abstract approach based upon the following fact:

CLAIM: *If $A \subset \mathbb{R}$ has an upper bound and is a closed subset, then the least upper bound of A belongs to that set,*

Before proving the claim, we indicate how it can be used to complete the last step of the argument in the problem. Since X is compact, it follows that $h[X] \subset \mathbb{R}$ is also compact and hence is closed and bounded. By the Claim, this means that the least upper bound b of $h[X]$ lies in that set. But this means that if $b = f(x_0)$, then $d(f(x_0), y_0) \geq d(f(x), y_0)$ for all $x \in X$.■

Proof of the Claim. Suppose that the least upper bound b does not belong to A. By the definition of least upper bound, we can find a sequence of points $x_n \in X$ such that $d(f(x_n), y_0) > b - \frac{1}{n}$. Since the left hand side is at most b, the Squeeze Principle for limits impolies that the limit of the sequence $d(f(x_n), y_0)$ must be b, and since A is closed we must have $b \in A$ — a contradiciton. The source of the contradiction was our assumption that $b \notin A$, so this must be false, and we are forced to conclude that b must belong to A.■

(X8) Since continuous real valued maps on connected spaces have the intermediate value property and $f(x) = d(u, x)$ is continuous, it follows that f takes every value between $0 = f(u)$ and $f(v) = d(u, v)$. One of these values is $\frac{1}{2}\, d(u, v)$.■

Hint for finding the examples $B \subset A \subset X$ related to Chapter 10*:* There are examples for which $X = \mathbb{R}$ and the subsets B and A are intervals of various types (closed, open or half-open); consider cases where A and B are different types of intervals.