Math 145A
Winter 2017

UPDATED GENERAL INFORMATION — MARCH 10, 2017

The second in-class examination

This examination will take place on Friday, March 17. It will consist of four questions, and it will cover material beginning with `math145Anotes06b.pdf` and continuing through the end of the file `math145Anotes13c.pdf`. All the problems will be explanations or proofs; the former is likely to include questions about how the abstract definitions apply to subsets of the real line or coordinate plane, and one or more will be taken directly from problems in the `aabUpdate` files (either this one, earlier ones, or both).

As usual, the examination will involve knowledge of definitions, basic results, their meanings in simple cases, some relatively uncomplicated logical derivations or proofs, and applications to specific questions which arise in single variable calculus. The assigned questions in the exercises, the contents of the examinations from the Winter 2014 and 2016 courses, and the questions in the review for Quiz 3 (see `aabUpdate07.145A.w17.pdf` and `aabUpdate08.145A.w17.pdf`) are good sources of practice problems. Here are some further items for study:

(1) Suppose that X is a topological space and every nonempty open subset $U \subset X$ contains at least two points. Prove that every point of X is a limit point of X.

(2) Let $A \subset X$ and $B \subset Y$, where X and Y are topological spaces. Prove that the limit point sets of A, B and $A \times B$ satisfy $L(A) \times L(B) \subset L(A \times B)$. As usual, take the product topology on $X \times Y$.

(3) Let A, B, X, Y be as in the previous exercise where both X and Y are nonempty, and assume that $A \times B$ is an open subset of $X \times Y$ with respect to the product topology. Prove that A and B are open in X and Y respectively. [*Hints:* Recall that the vertical and horizontal slices $\{x_0\} \times Y$ and $X \times \{y_0\}$ are homeomorphic to Y and X respectively by the maps sending (x_0, y) and (x, y_0) to y and x. Also, recall the definition of subspace topologies for vertical and horizontal slices.] — Why is a similar conclusion true if "open" is replaced by "closed"?

(4) Explain why the set of rational numbers x such that $|x| < \sqrt{2}$ is a closed subset of the rationals $\mathbb{Q}$ in the subspace topology.

(5) Let X be a space in which every one point subset is closed in X. Given a point $p \in X$, let $\mathcal{N}_p$ be the family of all open subsets in X containing p. If p and q are different points of X, why are $\mathcal{N}_p$ and $\mathcal{N}_q$ distinct families of subsets?

(6) Given a continuous function $f : X \to Y$, give an example such that X is Hausdorff but $f[X]$ is not.

(7) Let A be the subset of the coordinate plane consisting of all (x, y) such that either $x \geq 0$ or $y \geq 0$. Explain why A is connected.

(8) Describe all connected subsets A of the real line for which the interior of A is the open interval $(0, 1)$.

(9) Explain why the set of points (x, y) in the coordinate plane satisfying $|x| + |y| \leq 1$ is compact. [*Hint:* Why must we have $|x|, |y| \leq 1$, and why is the subset closed?]

(10) Explain why the set of points (x, y) in the coordinate plane satisfying $|x| \cdot |y| \leq 1$ is not compact.

(11) Let A be the set of all numbers $y = (3x + 4)/(5x + 6)$ where x ranges over all **positive** real numbers. What is the greatest lower bound of A? [*Hint:* Try graphing the function.]

(12) Let A be the set of all points (x, y) in the coordinate plane such that $y^2 = \pm 1$. Explain why A is not connected.

(13) Let A and B be bounded subsets of a metric space X. Explain why $A \cup B$ is also bounded, and if $A \cap B$ is nonempty give an upper estimate for its diameter involving the diameters of A and B. Also, give examples where this inequality fails if $A \cap B$ is empty.

(14) Let $f : X \to Y$ be a continuous function where X is compact and the topology on Y comes from some metric d. Explain why $f[X]$ is bounded, and if $y_0 \in Y$ explain why there is some point $x_0 \in X$ such that the distance function $d(f(x), y_0)$ takes a maximum value.

(15) Suppose that (X, d) is a connected metric space and u, v are distinct points of X. Prove that there is some point w such that $d(u, w) = \frac{1}{2}\, d(u, v)$.

(16) Suppose that (X, d) is a bounded connected metric space with diameter Δ. Prove that every number in $[0, \Delta)$ is the distance between two points of X.

(17) Let $A \subset \mathbb{R}^n$ be a bounded subset. Prove that every continuous real valued function $f : A \to \mathbb{R}$ is bounded and has a maximum value if and only if A is compact.

(18) Prove that there is a point $x \in \left(\frac{1}{2}\pi, \frac{3}{2}\pi\right)$ such that $\tan x = x$. [*Hint:* Graph the functions $f(x) = \tan x$ and $f(x) = x$. What are the limits of $\tan x - x$ as $x \to \frac{1}{2}\pi$ from the right and $x \to \frac{3}{2}\pi$ from the left?]

For Chapters 12 and 13, it is worthwhile to find examples where the criteria for recognizing connected and compact subspaces apply.

As noted before, *there will not be any course activity during the week of March* 20.

Course and examination grades

After the second examination has been graded, solutions and the grading curve will be posted for those who are interested in seeing them. My general policy is that students are welcome to take and keep their examinations. Students who want to retrieve their examinations should contact me by electronic mail next quarter so that arrangements can be made.