Comparing the metric and Zariski Topologies

Let $\mathbb{F}$ denote the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. In the file `math145Anotes07.pdf` we defined the **Zariski topology** on $\mathbb{F}^n$ whose closed subsets are the *affine varieties* $V(S)$, where S is some subset of the polynomial ring $\mathbb{F}[t_1, \cdots, t_n]$ in n indeterminates with coefficients in $\mathbb{F}$. Specifically, $V(S)$ consists of all points $a = (a_1, \cdots, a_n) \in \mathbb{F}^n$ such that $p(a) = 0$ for all $p \in S$. The open subsets for the Zariski topology are then the sets of the form $\mathbb{F}^n - V(S)$, and the arguments in the previously cited notes show that these sets form a topology on $\mathbb{F}^n$. More generally, if $\mathbb{F}$ is an arbitrary field, one can define the Zariski topology similary and prove that the associated open sets for a topology on $\mathbb{F}^n$.

The main result in this document describes the relationship between the Zariski topology and the usual metric topology on $\mathbb{F}^n$:

THEOREM 1. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let $\mathcal{T}_Z$ denote the Zariski topology on $\mathbb{F}^n$, and let $\mathcal{T}_M$ denote the metric topology on $\mathbb{F}^n$. Then $\mathcal{T}_Z$ is properly contained in $\mathcal{T}_M$.*

The following result, which is a key step in the proof of Theorem 1, is interesting and useful in its own right.

THEOREM 2. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.*

(i) If $p(t_1, \cdots, t_n)$ is a polynomial form in $\mathbb{F}[t_1, \cdots, t_n]$, then $V(p)$ is a closed subset of $\mathbb{F}^n$ with respect to the metric topology. Furthermore, if $V(p)$ contains a nonempty open subset, then p is the zero polynomial.

(ii) If $p(t_1, \cdots, t_n)$ is a nonzero polynomial form in $\mathbb{F}[t_1, \cdots, t_n]$, then the complement $\mathbb{F}^n - V(p)$ of $V(p)$ is an open dense subset of $\mathbb{F}^n$.

Proof. (i) The first conclusion follows because p is continuous, so that the inverse image $V(p)$ of the closed set $\{0\} \subset \mathbb{F}^n$ is a closed set.

We shall prove the second conclusion by induction on the number of indeterminates in the polynomial. The result is for polynomials in one indeterminate because a nontrivial polynomial in one variable has only finitely many roots. Assume it is true for polynomials of with $n-1$ indeterminates, where $n \geq 2$, and write the polynomial in the form

$$p(t_1, \cdots, t_n) \quad = \quad \sum_{j=0}^{d} q_j(t_1, \cdots, t_{n-1})\, t_n^j$$

where we might as well assume that $d > 0$ (otherwise we have a polynomial not involving the indeterminate t_n and the conclusion of the proposition follows from the induction hypothesis).

Suppose now that $p = 0$ on some open subset U, let $a = (a_1, \cdots, a_n) \in U$, and choose $h > 0$ such that the product open set

$$\prod_{i=1}^{n} N_h(a_i) \quad \subset \quad U \ .$$

If $(x_1, \cdots, x_n) \in U$, then by the preceding sentence we have

$$p(x_1, \cdots, x_n) \quad = \quad \sum_{j=0}^{d} q_j(x_1, \cdots, x_{n-1})\, x_n^j \ .$$

Then for each fixed choice of $(x_1, \cdots, x_{n-1}) \in \prod_{i=1}^{n-1} N_h(a_i)$, the polynomial

$$f(t_n) \quad = \quad p(x_1, \cdots, t_n) \quad = \quad \sum_{j=0}^{d} q_j(x_1, \cdots, x_{n-1})\, t_n^j$$

is zero whenever $t_n \in N_h(a_n)$, and since the proposition is known for polynomials in one indeterminate this polynomial must be zero. Therefore we know that $q_j(x_1, \cdots, x_{n-1}) = 0$ for all j and all $(x_1, \cdots, x_{n-1}) \in \prod_{i=1}^{n} N_h(a_i)$. We can now apply the induction hypothesis to conclude that q_j is the zero polynomial for each j, and this in turn implies that $p = 0$.■

(ii) Let U be a nonempty metrically open open subset of $\mathbb{F}^n$. Then by the first part of the result the open set U is not contained in $V(p)$, which means that $U - V(p)$ is not empty. Since this is one definition or characterization of a dense subset, the conclusion of (ii) follows.■

COROLLARY 3. *If bbF $= \mathbb{R}$ or $\mathbb{C}$, then every Zariski open set is also open and dense in the metric topology.*

The proof of this corollary requires some algebraic input:

HILBERT BASIS THEOREM FOR $\mathbb{F}[t_1, \cdots, t_n]$. *Let $\mathbb{F}$ be a field, let $\mathbb{F}[t_1, \cdots, t_n]$ denote the ring of polynomials in n indeterminates, and let J be an ideal in $\mathbb{F}[t_1, \cdots, t_n]$. Then there is a finite set of polynomials $\{p_1, \cdots, p_n\} \subset J$ such that every polynomial in J can be expressed as a linear combination*

$$\sum_j a_j\, p_j$$

for some polynomials a_j in n indeterminates.

One reference for a proof of the Hilbert Basis Theorem is Hungerford, *Algebra*, Theorem VIII.4.9, page 391 (the second example on page 372 shows that $\mathbb{F}$ is an example of a noetherian ring).

Proof of Corollary 3. If p is a polynomial then by continuity we know that $\mathbb{F}^n - V(\{p\})$ is open and dense by the theorem. By Additional Exercise 6.5.(ii) in `exercises02w14.pdf` we know that an intersection of two open and dense subsets is open and dense in the metric topology, so by a standard induction argument the same is true for a finite intersection of such sets. Since

$$V(\{p_1, \cdots, p_k\}) \quad = \quad \bigcap_j V(\{p_j\})$$

it follows that the right hand side is open and dense in the metric topology. Therefore, if we can show that each set $V(S)$ is equal to $V(T)$ for some finite set T, then the corollary will follow.

Given a commutative ring with unit R and a subset $S \subset R$, let $\langle S \rangle$ denote the ideal generated by S; in other words, $\langle S \rangle$ is the set of all finite linear combinations $\sum_\alpha b_\alpha\, s_\alpha$ where $s_\alpha \in S$ and $B_\alpha \in R$ for all α. If $R = \mathbb{F}[t_1, \cdots, t_n]$ then it follows immediately that $V(S) = V(\langle S \rangle)$. By the

Hilbert Basis Theorem an ideal $\langle S \rangle \subset \mathbb{F}[t_1, \cdots, t_n]$ is equal to $\langle T \rangle$ for some finite set T, and by the preceding paragraph this suffices to complete the proof of the corollary.■

Proof of Theorem 1. By Theorem 2 we know that $\mathcal{T}_Z$ is contained in $\mathcal{T}_M$, and that a set which is open in the Zariski topology is also dense in the metric topology. Therefore a metrically open subset will not be open in the Zariski topology if it is not dense. There are many such open subsets of $\mathbb{F}^n$; in particular, if $a \in \mathbb{F}^n$ and $r > 0$ then $N_r(a)$ is metrically open but not dense. This shows that the containment of topologies is proper.■

Final remark. The Zariski topology is not metrizable because the intersection of two nonempty open subsets is nonempty. In contrast, a metrizable space with more than two points always has a pair of nonempty disjoint subsets; specifically, if $x \neq y$ and $r = d(x, y)$, then $N_{r/2}(x)$ and $N_{r/2}(y)$ are disjoint (see Chapter 11 of Sutherland for more information on this topic).■