EXERCISES FOR MATHEMATICS 145B

SPRING 2015 — Part 1

The references denote sections of the text for the course:

J. R. Munkres, *Topology* (Second Edition), Prentice-Hall, Saddle River NJ, 2000. ISBN: 0–13–181629–2.

Solutions to nearly all these exercises are given in the `solutions` files (see the course directory). Other web sites with solutions to exercises in Munkres (including some not given in our files) are `http://www.math.ku.dk/∼moller/e03/3gt/3gt.html` and `http://dbfin.com/?=munkres` (but there are no guarantees that everything on these sites is free of errors).

0.6 : Components

(Munkres, §§ 23, 25)

Additional exercises

1. (*i*) Let X and Y be topological spaces. Prove that every connected component of $X \times Y$ has the form $C \times D$, where C is a connected component of X and D is a connected component of Y. [*Hint:* If K is a connected component of the product, consider $p_X[X]$ and $p_Y[Y]$ where p_X and p_Y are the coordinate projections onto X and Y.]

(*ii*) Prove an analogous result in which "arc component" replaces "connected component."

2. Let X be a connected space, and let $\mathcal{R}$ be an equivalence relation that is locally constant (for each point x all points in some neighborhood of x lie in the $\mathcal{R}$-equivalence class of x). Prove that $\mathcal{R}$ has exactly one equivalence class.
Note. See `zerogradient.pdf` for a simple but important consequence of this exercise.

3. Let U be an open subset of $\mathbb{R}^n$ for some n. Prove that U is connected if and only if U is arcwise connected. [*Hint:* If $\mathcal{A}$ is the equivalence relation whose equivalence classes are the arc components of U, why are these equivalence classes open in U? And why are they also closed in U?]

4. Prove that the closed annulus (ring shaped region) in $\mathbb{R}^2$ defined by $1 \leq x^2 + y^2 \leq 2$ is arcwise connected.

5. For each $n \geq 2$ prove that the unit sphere

$$S^{n-1} \quad = \quad \{ \ v \in \mathbb{R}^n \mid |v|^2 \quad = \quad 1 \ \}$$

and $\mathbb{R}^n - \{\mathbf{0}\}$ are arcwise connected. [*Hint:* Do the second one first.]

I. Complete metric spaces

I.1 : Definitions and basic properties

(Munkres, §§ 43, 45, 46)

Munkres, § 43, pp. 270–271: 1, 4, 6*bc*

Additional exercises

1. Prove that a discrete metric space is complete.

2. Show that the Nested Intersection Property for complete metric spaces (Munkres, Exercise 43.4) does not necessarily hold for nested sequences of closed subsets $\{ A_n \}$ if $\lim_{n\to\infty} \mathrm{diam}(A_n) \neq 0$; *i.e.*, in such cases one might have $\cap_n A_n = \emptyset$. [*Hint:* Consider the set A_n of all continuous functions f from $[0,1]$ to itself that are zero on $\left[\frac{1}{n}, 1\right]$ and also satisfy $f(0) = 1$.]

3. Two metrics d and d' on a set X are said to be *Lipschitz equivalent* if there are positive constants m and M such that

$$m \cdot d'(x,y) \quad \leq \quad d(x,y) \quad \leq \quad M \cdot d'(x,y)$$

for all $x, y \in X$. Show that if d and d' are Lipschitz equivalent, then X is complete with respect to d if and only if it is complete with respect to d'. [*Hint:* Explain why we also have the inequalities

$$M^{-1} \cdot d(x,y) \quad \leq \quad d'(x,y) \quad \leq \quad m^{-1} \cdot d(x,y)$$

for all $x, y \in X$.]

I.2 : The Contraction Lemma

(Munkres, §§ 43, 45, 46)

Additional exercises

1. Prove that the equation $x^5 + 7x - 1 = 0$ has a unique root in the interval $[0,1]$. [*Hint:* If $f(x) = (1 - x^5)/7$, then $x^5 + 7x - 1 = 0$ if and only if $f(x) = x$. Verify that f defines a function from $[0,1]$ to itself which satisfies the hypotheses of the Contraction Lemma.]

2. Give an example to show that the Contraction Lemma does not extend to self-maps of complete metric spaces such that $d(f(x), f(y)) \leq d(x,y)$ for all x and y. [*Hint:* There are standard examples for $\mathbb{R}$ such that $d(f(x), f(y)) = d(x,y)$ for all x and y, and one can even take f to be a linear function.]

3. Let X be a metric space. A map $f : X \to X$ is said to be a *proper similarity* of X if f is onto and there is a positive constant $C \neq 1$ such that

$$d\left(f(u),\, f(v) \right) \;=\; C \cdot d(u,v)$$

for all u, $v \in X$ (hence f is 1–1 and uniformly continuous). Prove that every proper similarity of a complete metric space has a unique fixed point. [*Hint and comment:* Use the Contraction Lemma if $C < 1$. If $C > 1$, let $g = f^{-1}$ and verify that

$$d\left(g(u),\, g(v) \right) \;=\; C^{-1} \cdot d(u,v)\ .$$

Why does $f(x) = x$ hold if and only if $g(x) = x$? If $X = \mathbb{R}^n$ and a similarity is given by $f(x) = cAx + b$ where A comes from an orthogonal matrix and $c \neq 1$ is a positive constant, then one can prove the existence of a unique fixed point directly using linear algebra; the crucial point is that 1 will not be an eigenvalue for the matrix cA.]

I.3 : Completions

(Munkres, §§ 43, 45, 46)

Additional exercises

1. Let X be a metric space. Prove that a union of two complete subspaces in X is also complete, and prove that a nonempty intersection of complete subspaces in X is also complete. [*Hint:* View X as a subspace of its completion.]

2. Suppose that X is a metric space and A is a dense subset (*i.e.*, $\overline{A} = X$. Let Y be a completion of X. Prove that A is also dense in Y and hence Y is also a completion of A. [*Hint:* Show that A is dense in Y; in other words, for each $y \in Y$ and $\varepsilon > 0$ there is some $a \in A$ such that $d(a, y) < \varepsilon$.]