EXERCISES FOR MATHEMATICS 145B

SPRING 2015 — Part 4

The remarks at the beginning of Part 1 also apply here. The references denote sections of the texts for the course (Munkres and Crossley).

IV. Homotopy groups

IV.1 : Pointed spaces

(Munkres, § 51; Crossley, §§ 8.1–8.2, 8.5)

Additional exercises

1. (a) If (X, x) is a pointed space and $A \subset X$, can we find a base point a for A such that the inclusion $(A, a) \to (X, x)$ is a base point preserving map? Give reasons for your answer.

(b) If (A, a) is a pointed space and $A \subset X$, why is it always possible to find a base point for X such that the inclusion map from A to X is base point preserving?

2. (a) Let (X, x) and (Y, y) be pointed spaces, and make $X \times Y$ into a pointed space by taking the base point to be (x, y). As usual let p_X and p_Y denote the coordinate projections onto X and Y respectively. Prove that if (W, w) is a pointed space, then a continuous mapping $f : W \to X \times Y$ is base point preserving if and only if $p_X \circ f$ and $P_Y \circ f$ are.

(b) Suppose that we are given the setting of (a), and assume that the sets $\{x\}$ and $\{y\}$ are closed in X and Y respectively. Define the **wedge** or **one point union** $(X, x) \vee (Y, y)$ to be the subspace

$$X \vee Y \quad = \quad X \times \{y\} \ \cup \ \{x\} \times Y \quad \subset X \times Y$$

with base point (x, y) as before. Prove that the pointed space $(X \vee Y, (x, y))$ has the following *universal mapping property:* If (Z, z) is a pointed space and $f : (X, x) \to (Z, z)$ and $g : (Y, y) \to (Z, z)$ are base point preserving maps, then there is a unique continuous base point preserving map $h : (X \vee Y, (x, y)) \to (Z, z)$ such that the restrictions of h to (X, x) and (Y, y) are f and g respectively.

3. A continuous mapping $i : A \to X$ is said to be a **retract** if there is a continuous mapping $r : X \to A$ such that $r \circ i$ is the identity on A.

(a) Prove that the mapping i is 1–1 and the mapping r is onto.

(b) If X is Hausdorff, prove that i is a closed mapping (and hence the topology on A is the subspace topology).

(c) If $a \in A$ is a base point, show that there is a base point for X such that both i and r are base point preserving maps.

(d) Suppose that $i : A \to X$ is a retract and Y is an arbitrary space. Let $f, g : Y \to A$ be continuous mappings. Prove that if $i \circ f \simeq i \circ g$, then $f \simeq g$. [*Hint:* What happens if we compose with r?]

4. (*a*) Let P be the one point space $\{q\}$, and take q to be its base point (in fact, there is only one possible choice). Given a pointed space (X, x), explain why there are unique continuous base point preserving mappings $(P, q) \to (X, x)$ and $(X, x) \to (P, q)$.

(*b*) Suppose that (N, n) is a pointed space such that for each pointed space (X, x) there are unique continuous base point preserving mappings $(N, n) \to (X, x)$ and $(X, x) \to (N, n)$. Prove that N consists of a single point. [*Hint:* The hypotheses imply that there is a unique continuous base point preserving mapping from (N, n) to itself. Why is this the identity mapping, and what does this imply about the composites $(N, n) \to (P, q) \to (N, n)$ and $(P, q) \to (N, n) \to (P, q)$?]

IV.2 : Algebraic Structure

(Munkres, § 52; Crossley, §§ 8.1–8.3)

Additional exercises

1. (*a*) Let X be a topological space, and suppose that α and β are continuous curves in X such that $\alpha(0) = \beta(0) = x_0$ and $\alpha(1) = \beta(1) = x_1$. Prove that α is endpoint preservingly homotopic to β if and only if $\alpha + (-\beta)$ is base point preservingly homotopic to a constant curve. Why is this also equivalent to the condition that $\beta + (-\alpha)$ is homotopic to a constant curve?

(*b*) Let X be an arcwise connected space. Prove that the following are equivalent:

For every $p, q \in X$, all continuous curves joining p to q are endpoint preservingly homotopic.

For every $x \in X$, all continuous closed curves based at x are base point preservingly homotopic.

A space satisfying either (hence both) of these conditions is said to be *simply connected*.

2. If K is a nonempty convex subset of $\mathbb{R}^n$, explain why K is simply connected.

3. Suppose that (X, x) is a Hausdorff space, let $x \in A \subset X$, and suppose that the inclusion map $i : (A, x) \to (X, x)$ is a retract with one-sided inverse $r : (X, x) \to (A, x)$. Prove the induced map of fundamental groups i_* is 1–1, and also prove that every element of $\pi_1(X, x)$ has a unique factorization of the form $i_*(u) \cdot v$, where $u \in \pi_1(X, x)$ and v lies in the kernel of r_*.

WARNING. $(i_*(u_1) \cdot v_1) \cdot (i_*(u_2) \cdot v_2)$ is necessarily not equal to $i_*(u_1 u_2) \cdot v_1 v_2$.

IV.3 : Simple cases

(Munkres, § 54; Crossley, §§ 6.3, 8.3)

Munkres, § 52, pp. 334–335: 1*b*, 5, 7

Additional exercises

1. Let $f : (S^1, 1) \to (S^1, 1)$ be a continuous mapping which is not onto. Prove that f represents the trivial element of $\pi_1(S^1, 1)$. [*Hint:* Let $g : [0, 1] \to \mathbb{R}$ be the unique lifting such that $g(0) = 0$; more precisely, we have $f(\cos 2\pi\, t, \sin 2\pi\, t) = (\cos 2\pi\, g(t), \sin 2\pi\, g(t)))$. Why is f onto if $g(1) \neq 0$? Explain why if $g(1) \neq 0$ then the image of g contains either $[-1, 0]$ or $[0, 1]$.]

2. Identify the space of $n \times n$ matrices over the complex numbers with $\mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$, and let $\mathbf{GL}(n,\mathbb{C})$ denote the group of invertible $n \times n$ matrices over the complex numbers. Let $j : S^1 \to \mathbf{GL}(n,\mathbb{C})$ denote the homomorphism which sends $z \in S^1$ to the diagonal matrix whose upper left entry is z and whose remaining diagonal entries are ones. Prove that j induces a 1–1 mapping of fundamental groups. [*Hint:* If $\Delta : \mathbf{GL}(n,\mathbb{C}) \to \mathbb{C} - \{0\}$ is the determinant map, why is Δ continuous and why is $\Delta \circ j$ a homotopy equivalence?]

IV.4 : Change of base point

(Munkres, § 52)

Munkres, § 54, pp. 334–335: 6

Additional exercises

1. Compute the fundamental group of $(\mathbb{R}^2 - S^1, z)$ for all choices of $z \in \mathbb{R}^2 - S^1$.

2. Let G be an arcwise connected topological group with identity element 1, and let $g \in G$. Why do all continuous paths from 1 to g induce the same isomorphism from $\pi_1(G,1)$ to $\pi_1(G,g)$?

Definitions. Let $p : E \to B$ be a continuous mapping. The p is said to have the Path Lifting Property (**PLP**) if the first statement below is true, and p is said to have the Covering Homotopy Property (**CHP**) if the second statemnt below is true:

> (**PLP**) *Let $b_0 \in B$, and assume that $p(e_0) = b_0$. Then a continuous path $f : [0,1] \to B$ beginning at b_0 has a unique lifting to a continuous path $f : [0,1] \to E$ beginning at e_0.*
>
> (**CHP**) *Let b_0 and e_0 be as above, and assume that $h : [0,1] \times [0,1] \to B$ is continuous with $h(0,0) = b_0$. Then there is a unique lifting of h to a continuous map $H : [0,1] \times [0,1] \to E$ such that $H(0,0) = e_0$.*

The results of Section III.3 in the course imply that the map $p : \mathbb{R} \to S^1$ given by $p(t) = \exp(2\pi\, i\, t)$ has both of these properties.

3. (*a*) Prove that if p has the Path Lifting Property and B is arcwise connected, then p is onto.

(*b*) Prove that if p has the Covering Homotopy Property, then p also has the Path Lifting Property. [*Hint:* Consider the homotopy $H(s,t) = \gamma(s)$.]

4. (Compare Munkres, Theorems 53.2 and 53.3.) (*a*) Prove that if $p : E \to B$ and $p' : E' \to B'$ satisfy the PLP and CHP, then so does $p \times p' : E \times E' \to B \times B'$.

(*b*) Suppose that $p : E \to B$ satisfies the PLP and CHP and B_0 is a subspace of B. If $E_0 = p^{-1}[B_0]$, then the map $p_0 : E_0 \to B_0$ obtained by restricting p also satisfies the PLP and CHP.

5. The purpose of this exercise is to prove the existence of a (pointed) space (X,x) such that $\pi_1(X,x)$ is not abelian.

(*a*) Suppose that $(X,x) = (S^1 \vee S^1, (1,1))$ as in Exercise IV.1.2, so that X is a union of the two circles $S^1 \times \{1\}$ and $\{1\} \times S^1$ which have a single point in common. Using the preceding exercise, prove that the restriction of $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ to the inverse image of X defines a continuous mapping

$$q : \Gamma \ = \ (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}) \ \longrightarrow \ X$$

which satisfies the PLP and CHP (see `exercises4as15.pdf` for a drawing of Γ, which is a union of vertical and horizontal lines).

(*b*) For $j = 0, 1$ let $i_j : (S^1, 1) \to (S^1, 1) \times (S^1, 1)$ denote the base point preserving slice inclusion into the j^{th} factor, and let $\theta : [0, 1] \to S^1$ be the usual counterclockwise parametrization $\theta(t) = \exp(2\pi\, i\, t)$, and define θ_j to be the composite $i_j \circ \theta$. Explain why the unique lifting Δ of the iterated concatenation $(\theta_1 + \theta_2) + ((-\theta_1) + (-\theta_2))$ is the usual counterclockwise parametrization for the boundary F of the unit square; namely, $(\alpha_1 + \beta_2) + +((-\alpha_2) + (-\beta_1))$, where $\alpha_1(t) = (t, 0)$, $\alpha_2(t) = (t, 1)$, $\beta_1(t) = (0, t)$ and $\beta_2(t) = (1, t)$. You do not need to give a detailed argument.

(*c*) Show that the parametrization for the boundary F of the unit square in (*b*) defines a homeomorphism from S^1 to F.

(*d*) Show that the closed curve Δ is not homotopic to the trivial element of $\pi_1(\Gamma, (0, 0))$ as follows: First, explain why (*c*) implies that Δ does not represent the trivial element of $\pi_1(F, (0, 0))$. Next, prove that F is a retract of Γ by considering the composite of

(1) the map $\Gamma \to ([0, 1] \times \mathbb{Z}) \cup (\{0, 1\} \times \mathbb{R})$ sending (u, v) to itself if $u \in [0, 1]$, to $(0, v)$ if $u \leq 0$ and to $(1, v)$ if $u \geq 1$,

(2) the map $([0, 1] \times \mathbb{Z}) \cup (\{0, 1\} \times \mathbb{R}) \to F$ sending (u, v) to itself if $v \in [0, 1]$, to $(u, 0)$ if $v \leq 0$ and to $(u, 1)$ if $v \geq 1$.

In other words if r is the composite of the first map followed by the second, why is $r|F$ the identity? The drawings in the file `exercises4as15.pdf` might be helpful.

(*d*) Use the CHP to show that if the image of Δ represents the trivial element of the group $\pi_1(S^1 \vee S^1, (1, 1))$, then Δ also represents the trivial element in $\pi_1(\Gamma, (0, 0))$.

(*e*) Why does the image of Δ in $\pi_1(S^1 \vee S^1, (1, 1))$ represent the monomial

$$[\theta_1]\,[\theta_2]\,[\theta_1]^{-1}\,[\theta_2]^{-1}$$

and why does this imply that the fundamental group of the wedge is nonabelian?

(*f*) Explain why the image of Δ in $\pi_1(S^1 \times S^1, (1, 1))$ is trivial.