EXERCISES FOR MATHEMATICS 145B

SPRING 2015 — Part 5

The remarks at the beginning of Part 1 also apply here. The references denote sections of the texts for the course (Munkres and Crossley).

V. Further topics

V.1 : Homotopy and line integrals

(Munkres, §§ 65, 66)

Additional exercises

1. (a) Let $U \subset \mathbb{R}^2$ be the set of all points (x, y) such that $x > 0$, and let γ be a regular piecewise smooth curve in U. Explain why

$$\int_\gamma \frac{x\,dy \;-\; y\,dx}{x^2 + y^2}$$

is equal to zero without computing any integrals explicitly. [*Hint:* What is the fundamental group of U?]

(b) Let γ be the usual counterclockwise unit circle in $\mathbb{R}^2 - \{\mathbf{0}\}$, and let λ be a second curve in the same open set such that $|\gamma(t) - \lambda(t)| < 1$ for all $t \in [0, 1]$. Prove that

$$\int_\gamma \frac{x\,dy \;-\; y\,dx}{x^2 + y^2} \quad = \quad \int_\lambda \frac{x\,dy \;-\; y\,dx}{x^2 + y^2} \ .$$

[*Hint:* Why does the image of the straight line homotopy lie in $\mathbb{R}^2 - \{\mathbf{0}\}$?]

2. Let $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ be a vector field on $\mathbb{R}^2 - \{\mathbf{0}\}$ such that P and Q have continuous partial derivatives which satisfy the condition $Q_x = P_y$ on partial derivatives, and assume also that $\int_\theta \mathbf{F} \cdot d\mathbf{x} = 0$ when θ is the standard counterclockwise parametrization of the unit circle.

(a) Prove that $\int_\gamma \mathbf{F} \cdot d\mathbf{x} = 0$ for all closed continuous rectifiable curves in $\mathbb{R}^2 - \{\mathbf{0}\}$. Also, prove that if α and β are two continuous rectifiable curves curves in $\mathbb{R}^2 - \{\mathbf{0}\}$ with the same endpoints, then

$$\int_\alpha \mathbf{F} \cdot d\mathbf{x} \quad = \quad \int_\beta \mathbf{F} \cdot d\mathbf{x} \ .$$

[The most basic examples of continuous rectifiable curves are regular piecewise smooth curves; what one needs is a simple condition to guarantee that the integrals in question are definable, and rectifiability is the standard abstract condition of this sort.]

(b) Using (a), prove that there is some function $g(x, y)$ on U such that $\nabla g = \mathbf{F}$. [*Hint:* Fix a point p in U and define g to $g(x, y)$ to be the the line integral $\int_\gamma \mathbf{F} \cdot d\mathbf{x}$ for a suitably well behaved curve joining p to (x, y); by (a) the value of this integral does not depend upon the choice of γ. Then show that the partial derivatives of g are P and Q).

3. Let U be a simply connected open subset of $\mathbb{R}^2$ (arcwise connected and trivial fundamental group), and let $\mathbf{F} = (P, Q)$ be a vector field on U where P and Q have continuous partial derivatives. Show that $\int_\gamma \mathbf{F} \cdot d\mathbf{x} = 0$ for all closed rectifiable curves in U if and only if

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

at all points of U. [*Hint:* If this condition does not hold everywhere, it fails on an open disk centered at some point, and in fact the difference is either always positive or always negative on a sufficiently small closed disk D centered at the point and contained in U. If Γ is the counterclockwise circle on the boundary of D, use Green's Theorem to show that $\int_\Gamma \mathbf{F} \cdot d\mathbf{x} \neq 0$.]

V.2 : Graph complexes

(Munkres, § 64; Crossley, §§ 7.1)

In all exercises for this unit, assume the graph is *simple* in the sense that no two edges have the same pair of endpoints.

Additional exercises

1. The edges of the k-dimensional hypercube $[0, 1]^n \subset \mathbb{R}^n$ form a connected graph whose vertices are the points of $\{0, 1\}^n$, with an edge joining two vertices if and only if they differ in exactly one coordinate. Show that this graph has 2^n vertices and $n\, 2^n$ edges.

2. (a) If $(X, \mathcal{E})$ is a graph with V vertices, explain why the number E of edges is at most $\binom{V}{2}$.

(b) If $(X, \mathcal{E})$ is a graph with V vertices and

$$E \geq \binom{V-1}{2}$$

show that X is connected.

3. A graph $(X, \mathcal{E})$ is said to be a **tree** if for each pair of vertices $p \neq q$ in X there is a unique reduced edge path joining p to q.

(a) Show that there is an edge structure $\mathcal{E}$ on $X = [0, 1]$ such that $(X, \mathcal{E})$ is a tree, and prove that for each n there is a tree such that at least one vertex lies on n distinct edges. [*Hint:* Stars.]

(b) Prove that a tree has no (reduced) circuits. [The converse is also true for connected graphs, but the proof is beyond the scope of this course.]

4. (a) Let $f : X \to Y$ be a homeomorphism of topological spaces, and let $E \subset X$ be a finite subset such that $X - E$ has k components for some nonnegative integer k. Prove that $Y - f[E]$ also has k components.

(b) Suppose that k is as above and n is a positive integer. Let $\mathbf{S}_{n,k}(X)$ be the set of all subsets E of X with n elements such that $X - E$ has k components. If $f : X \to Y$ is a homeomorphism, explain why $\mathbf{S}_{n,k}(X)$ and $\mathbf{S}_{n,k}(Y)$ have the same numbers of elements. [*Note:* This number may be infinite; consider $X = \mathbb{R}$ with $n = 1$ and $k = 2$. The number may also be zero; consider $\mathbb{R}$ with $n = 1$ and $k = 3$.]

5. (a) Use the numbers $\mathbf{S}_{n,k}(X)$ in the preceding exercise to write down a formal proof that the open interval $(0, 1)$, the closed interval $[0, 1]$ and the half open interval $(0, 1]$ are pairwise nonhomeomorphic.

(b) Consider the following depictions of some standard numerals and letters as connected graphs using sans-serif type:

4 6 8 0 A E I X

Using the numbers $\mathbf{S}_{n,k}$ in the preceding exercise, show that **8** distinct homeomorphism types are obtainable in this fashion. Are new homeomorphism types added if we include the remaining letters of the Latin alphabet? Explain. — Obviously, one can formulate similar questions for a more or less arbitrary set of printed characters. [*Comments:* It should suffice to use numbers $\mathbf{S}_{n,k}$ with $(n, k) = (1, a)$ or $(a, 1)$ where $a \leq 4$.]

6. Suppose that $(X, \mathcal{E})$ is a graph and $A \subset X$ is a finite subset. Prove that $X - A$ has only finitely many connected components. [*Hint:* First show that if E is an edge then $E - A$ has only finitely many connected components.]

V.3 : Chains, homology and fundamental groups

(Crossley, § 9.1)

Additional exercises

1. Let $(X, \mathcal{E})$ be a graph with vertices $\{v_1, \cdots, v_n\}$ and edges joining v_i to v_{i+1} for $1 \leq i < n$ and also joining v_n to v_1. Assume the vertices are ordered by their index numbers. For an arbitrary field $\mathbb{F}$ prove that $H_1(X, \mathcal{E}^\omega; \mathbb{F}) \cong \mathbb{F}$ and construct an explicit cycle representing a generator if $\mathbb{F} = \mathbb{Z}_3$.

2. Let $(X, \mathcal{E})$ be a graph with vertices $\{v_1, \cdots, v_n\}$. This graph is said to be *complete* if for every pair of vertices $\{v_i, v_j\}$ there is an edge $E_{i,j}$ joining v_i and v_j. Assuming that $(X, \mathcal{E})$ is a complete graph with n vertices, compute $H_i(X, \mathcal{E}^\omega; \mathbb{F})$, where ω is the linear ordering given by the indices v_j and $\mathbb{F}$ is an arbitrary field.

3. Let $(X, \mathcal{E})$ be a connected graph which is a union of two connected subgraphs $(X_1, \mathcal{E}_1)$ and $(X_2, \mathcal{E}_2)$ which have exactly one vertex in common. Prove that $H_1(X, \mathcal{E}^\omega; \mathbb{F})$ is isomorphic to a direct sum of $H_1(X_1, \mathcal{E}_1^\omega; \mathbb{F})$ and $H_1(X_2, \mathcal{E}_2^\omega; \mathbb{F})$; take the vertex orderings on the subgraphs which come from the vertex ordering on the original graph.

4. Let $(X, \mathcal{E})$ be the connected graph with vertices

$$a\ , \qquad b_i\ (1 \leq i \leq n)\ , \qquad c_j\ (1 \leq j \leq n)$$

and edges

$$ab_i\ (1 \leq i \leq n)\ , \qquad ac_j\ (1 \leq j \leq n)\ , \qquad b_k c_k\ (1 \leq k \leq n)\ .$$

In other words, it is the union of n triangles such that each pair meets at the vertex a. Show that $\dim H_1(X, \mathcal{E}^\omega; \mathbb{F}) = n$.

NOTE. Further results on graphs and fundamental results in homotopy theory imply that the underlying spaces of two connected (finite) graph complexes $(X_1, \mathcal{E}_1)$ and $(X_2, \mathcal{E}_2)$ are homotopy equivalent if and only if $\dim H_1(X_1, \mathcal{E}_1^{\omega_1}; \mathbb{F}) = \dim H_1(X_2, \mathcal{E}_2^{\omega_2}; \mathbb{F})$, and the exercise implies that each positive integer can be realized as the dimension of $\dim H_1(X, \mathcal{E}^{\omega}; \mathbb{F})$ for some connected graph $(X, \mathcal{E})$. Note that if $(X_0, \mathcal{E}_0)$ is the graph with $X = [0,1]$ and a single edge, and ω is the usual ordering of the endpoints, then $\dim H_1(X_0, \mathcal{E}_0^{\omega}; \mathbb{F}) = 0$, and therefore all nonnegative integers can be realized.

V.4 : Euler paths

(No textbook references)

Additional exercises

1. Show that if a connected graph $(X, \mathcal{E})$ has a closed Euler path, then $(X, \mathcal{E})$ has the same structure as a quotient of the graph in Additional Exercise V.2.1, in which the equivalence classes with more than one element are nonempty sets of vertices (look at `bowtie-graph.pdf` to see an example).

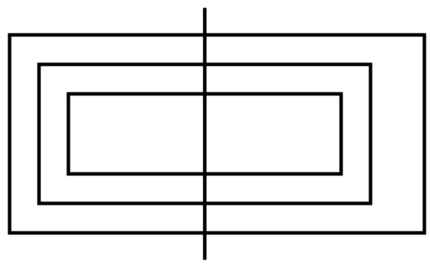
2. Is there a connected graph $(X, \mathcal{E})$ with a closed Euler path such that the number of vertices is even and the number of edges is odd? Either give an example or prove that no such graph can exist.

3. For each of the connected graphs on the next two pages, determine whether an Euler path exists, and construct an Euler path for the first case in which the answer is yes.

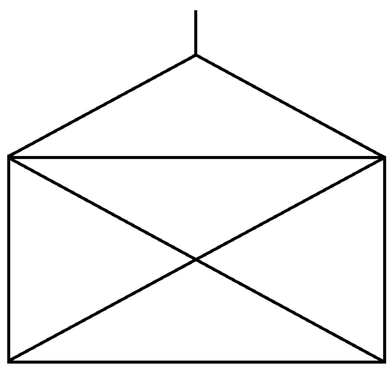
NOTE. There is a systematic method for finding an Euler path of this sort if the graph satisfies the condition in Theorem V.4.3 known as the *Fleury algorithm*. Details are given in the book by Bondy and Murty. For this exercise, it is only necessary to give an example if one exists.

Examples for Additional Exercise V.4.3

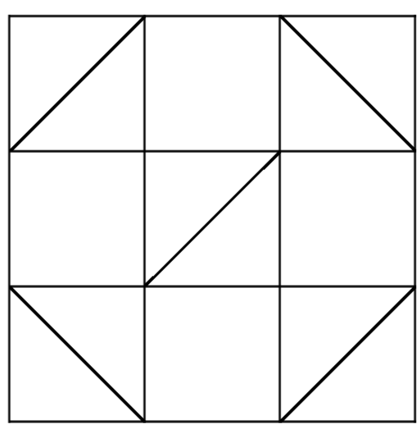
(*a*)



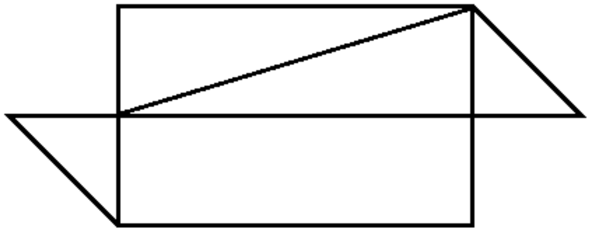
(*b*)



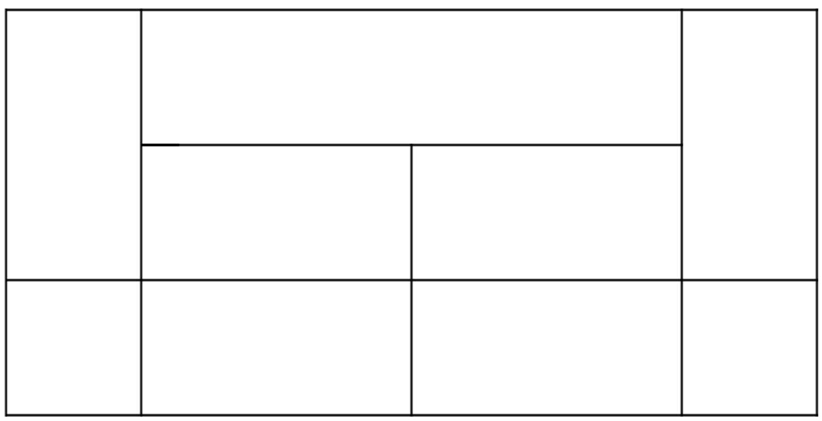
(*c*)



(*d*)



(*e*)



(*f*)