# 0.6 Components

Need some results from Section 23 of Munkres:

**Thm. 23.3** $\{A_\alpha\}$ a nonempty family of subsets in $X$, $x_0 \in \cap A_\alpha$. If each $A_\alpha$ is connected, then so is $\cup_\alpha A_\alpha$.

**Thm. 23.4** $A \subseteq X$ connected, $A \subseteq B \subseteq \overline{A} \Rightarrow B$ connected.

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**Definition** $X$ top. space. Define a binary operation $\mathcal{R}$ on $X$ by $x_0 \sim x_1 \Leftrightarrow$ there is a connected subset $C \subseteq X$ such that $x_0, x_1 \in C$.

**Lemma 0.6.0** $\mathcal{R}$ is an equivalence relation.

<u>Proof</u> $x_0 \mathcal{R} x_0$ — take $C = \{x_0\}$.

$x_0 \mathcal{R} x_1 \Rightarrow x_1 \mathcal{R} x_0$ — the defining condition is symmetric in $x_0$ and $x_1$.

$x_0 \mathcal{R} x_1$ & $x_1 \mathcal{R} x_2 \Longrightarrow x_0 \mathcal{R} x_2$ — If $A$ & $B$ are connected subsets such that $x_0, x_1 \in A$ and $x_1, x_2 \in B$, then $x_1 \in A \cap B \Rightarrow A \cup B$ is connected, with $x_0, x_2 \in A \cup B$.

by 22.3

<u>Proposition 0.6.1</u> The equivalence classes of $\mathcal{R}$ are maximal connected subsets of $X$, and these subsets are closed.

The equivalence classes are called the <u>connected components</u> of $X$.

<u>Proof.</u> Let $C$ be the equivalence class of $x_0 \in X$. We need to show $C$ is a maximal connected subset and $C$ is closed in $X$. The second statement will follow from the first and Munkres, Thm. 23.4, so we shall concentrate on proving the first one.

<u>Claim</u> $C$ is a union of connected subsets containing $x_0$. For $y \in C \Rightarrow$ there is some connected subset $A_y$ with $x_0, y \in A_y$. By definition $A_y \subseteq C$, and by Thm 23.3, $\cup A_y$ is connected. Since

$$C = \bigcup_{\substack{y \in C \\ y \neq X}} \{x_0, y\} \subseteq \cup A_y \subseteq C,$$

it follows that

$C = \cup A_y$, which is connected.

To see that $C$ is a maximal

connected subset, suppose that $C \subseteq D$ where $D$ is connected. Then $d \in D \Longrightarrow x_0 \mathcal{R} d$ by definition of $\mathcal{R}$, and thus $d \in C$, so that $D \subseteq C$ and hence $D = C$. ∎

<u>Note</u> If there are only finitely many components, each connected component is also open: If $C_1, \dots, C_n$ are the components then $X - C_i = \bigcup_{j \neq i} C_j$ is closed and hence $C_i$ is open.

However, if there are infinitely many components, then the latter need not be open. For example, in $\mathbb{Q}$ the connected components are one point sets, and no such subset is open in $\mathbb{Q}$.

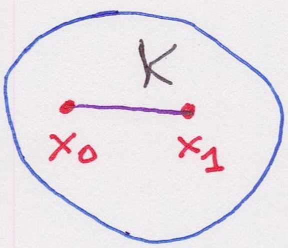
(every open interval in $\mathbb{Q}$ of the form $(q-h, q+h)$ is disconnected!).

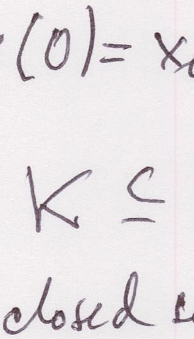
## Arcwise Connected Spaces

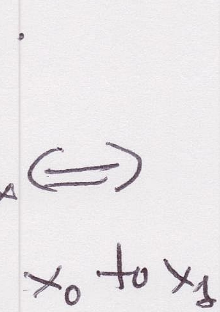
A useful variant of connectedness.

**Definition** A topological space is $\left\{\begin{array}{l}\text{path(wise)}\\ \text{arc(wise)}\end{array}\right\}$ connected if for each $x_0, x_1 \in X$ there is a continuous $\left\{\begin{array}{l}\text{curve}\\ \text{map}\end{array}\right\}$ $\gamma : [0,1] \longrightarrow X$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$.

**Examples** $K \subseteq \mathbb{R}^n$ is convex $\Longleftrightarrow$ $x_0, x_1 \in K \Longrightarrow$ closed segment joining $x_0$ to $x_1$ is contained in $K$.




"no dents 

or holes 

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neither is convex

<u>Proposition 0.6.2</u> A convex set is arcwise connected.

<u>Proof.</u> The straight line segment is parametrized by $(1-t)x_0 + tx_1 = \gamma(t)$. ∎

We shall give examples to show that some arcwise connected subsets of $\mathbb{R}^2$ are not convex, but first some generalities.

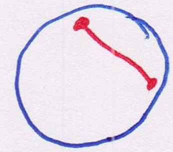
<u>Proposition 0.6.3</u> If $K_\alpha \subseteq \mathbb{R}^n$ is convex and $\cap K_\alpha \neq \phi$, then $\cap K_\alpha$ is convex.

nonempty family

<u>Proof.</u> $x_0, x_1 \in \cap K_\alpha \Rightarrow$ for each $\alpha^*$ we have $x_0, x_1 \in K_{\alpha^*}$, so that the closed segment $\{tx_1 + (1-t)x_0\} \subseteq K_{\alpha^*}$ for each $\alpha^*$ and hence it is contained in $\cap K_\alpha$. ∎

<u>Examples</u> $N_\varepsilon(p) \subseteq \mathbb{R}^n$ is convex,

$\overline{N_\varepsilon(p)} \subseteq \mathbb{R}^n$ is convex.



(i) $|x-p|, |y-p| < \varepsilon \Longrightarrow$ if $0 \le t \le 1$ then

$$\left|(1-t)x + ty - p\right| = \left|(1-t)(x-p) + t(y-p)\right| \le$$

$$|(1-t)(x-p)| + |t(y-p)| = (1-t)|x-p| +$$

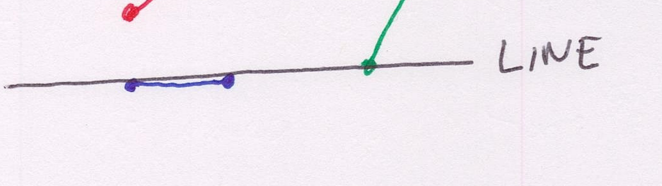
$$t|y-p| < (1-t)\varepsilon + t\varepsilon = \varepsilon.$$

(ii) Same, but replace $<$ with $\le$.

<u>More examples in $\mathbb{R}^n$</u>

The hyperplane $\sum a_i x_i = b$ is convex.

The half spaces $\sum a_i x_i \left\{\begin{matrix} \le \\ < \\ > \\ \ge \end{matrix}\right\} b$ are convex.



These yield many other examples; e.g., $[0,1]\times[0,1] \subseteq \mathbb{R}^2$ is given by

$$\{x \geq 0\} \cap \{x \leq 1\} \cap \{y \geq 0\} \cap \{y \leq 1\}.$$

<u>Verification for the closed half space.</u>

Suppose $x, y$ satisfy $\sum a_i x_i, \sum a_i y_i \geq b$.

Then $\sum a_i((1-t)x_i + ty_i) =$

$(1-t)\sum a_i x_i + t\sum a_i y_i \geq (1-t)b + tb = b.$

For the other cases, replace $\geq$ with $=, >, \leq, <$. ∎

<u>Generalities</u> $X$ top space. Write

$x_0 \mathcal{A} x_1 \iff$ there is a continuous $\gamma : [0,1] \longrightarrow X$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$.

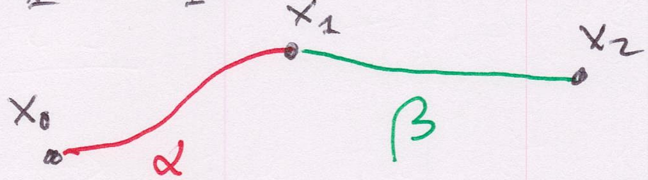
<u>Proposition 0.6.5</u> $\mathcal{A}$ is an equivalence relation, and the equivalence classes are the maximal arcwise connected subset. $\left(\begin{smallmatrix}\text{arc}\\ \text{path}\end{smallmatrix}\right)$ components of $X$.

<u>Proof.</u> Start with an equivalent formulation: $x_0 \mathcal{A} x_1 \Longleftrightarrow$ there is an arcwise connected subset $P \subseteq X$ such that $x_0, x_1 \subseteq P$. (Left as an exercise!)

$x_0 \sim x_0$ : Take the constant curve

$x_0 \sim x_1 \Rightarrow x_1 \sim x_0$ : If $\gamma$ joins $x_0$ to $x_1$ and $\beta(t) = \gamma(1-t)$, then $\beta$ joins $x_1$ to $x_0$.

$x_0 \sim x_1 + x_1 \sim x_2 \Rightarrow x_0 \sim x_2$




Concatenate curves to get $\alpha+\beta$, with

$$\alpha+\beta(t) = \begin{cases} \alpha(2t) & t \leq \frac{1}{2} \\ \beta(2t-1) & t \geq \frac{1}{2}. \end{cases}$$

It follows that the arc components are arcwise connected. To prove that they are maximal, let $P$ be an arc component, and let $P \subseteq P'$ where $P'$ is arcwise connected. Then $x \in P$ & $y \in P'$ $\Rightarrow x \mathcal{A} y \Rightarrow y \in P$. ∎

<u>Proposition 0.6.6</u> $f : X \to Y$ continuous, $X$ arcwise connected $\Rightarrow f[X]$ arcwise connected.

<u>Proof.</u> Suppose $y_0, y_1 \in f[X]$, $y_i = f(x_i)$. Let $\gamma$ join $x_0$ to $x_1$. Then $f \circ \gamma$ joins $y_0 = f(x_0)$ to $y_1 = f(x_1)$. ∎

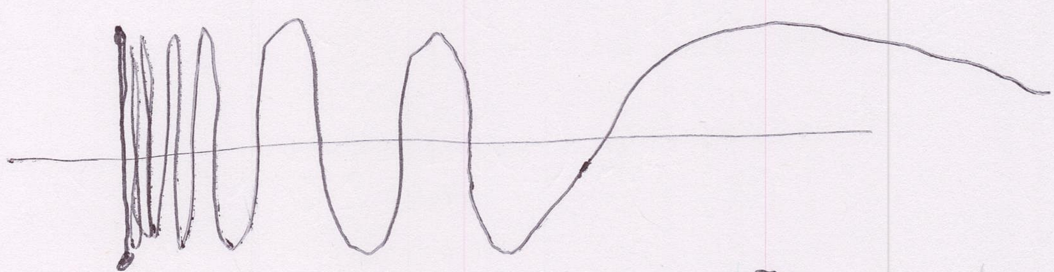
<u>Theorem 0.6.7</u> $X, Y$ arcwise connected $\Rightarrow$ so is $X \times Y$.

<u>Proof.</u> Let $(x_0, y_0), (x_1, y_1) \in X \times Y$. Let $\alpha$ join $x_0$ to $x_1$, $\beta$ join $y_0$ to $y_1$. Then $\gamma(t) = (\alpha(t), \beta(t))$ joins $(x_0, y_0)$ to $(x_0, y_1)$. ■

Recall $Z \to X \times Y$ is continuous $\Leftrightarrow$ its projections onto $X$ and $Y$ are continuous.

<u>EXAMPLES</u> Connected $\not\Rightarrow$ arcwise conn.

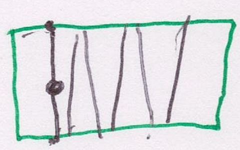
Arc components are not necessarily closed.



$$X = \{0\} \times [-1, 1] \cup \{(x,y) \in \mathbb{R}^2, x > 0 + y = \sin\tfrac{1}{x}\}.$$

The arc components are the graph and the vertical line segment. The first of these subsets is dense in $X$ and hence $X$ is connected. However the first subset is not closed in $X$.

<u>Key fact</u>: Every point $p$ in $\{0\}\times[-1,1]$ has an open neighborhood $U$ in $X$ such that the arc component of $p$ in $U$ is contained in $\{0\}\times[-1,1]$. — This means that if a curve starts in $\{0\}\times[-1,1]$ it stays in that subset.



<u>Example</u>: A small neighborhood of $(0,0)$ in $X$.

For details see pp. 67-8 of gentop-notes.pdf.