II.3 Scissors and paste theorems

<u>Theorem II.3.1</u> Let X be a topological space with $X = A \cup B$ where A and B are $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ in X. Let $\mathcal{R}$ be the equivalence relation on $A \amalg B$ generated by $(x,1) \sim (x,2)$ where $x \in A \cap B$. Then X is homeomorphic to $A \amalg B / \mathcal{R}$.

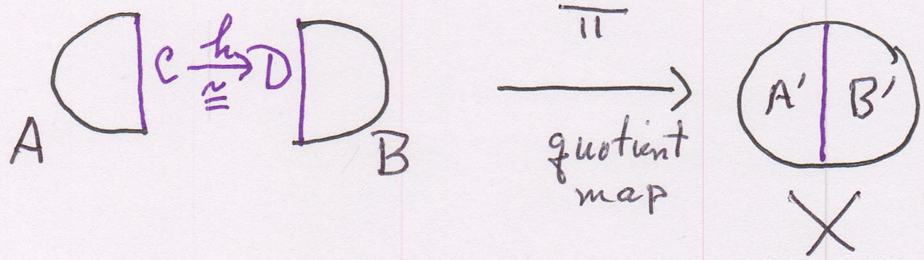
<u>PROOF.</u> Let i_A, i_B denote inclusions into X, and let $J_0 : A \amalg B \longrightarrow X$ be the continuous map given by i_A on $A \times \{1\}$ and i_B on $B \times \{2\}$. Then there is a unique continuous map $J : A \amalg B/\mathcal{R} \longrightarrow X$ induced by J_0. Since the eq. classes are one point sets $(a,1)$ with $a \in A - B$, $(b,2)$ with $b \in B - A$ and $\{(x,1),(x,2)\}$ with $x \in A \cap B$, it follows that J is 1-1 onto (& continuous).

II.3.2

To show that h is a homeomorphism, it suffices to show that $J_0 : A \amalg B \to X$ is $\begin{Bmatrix}\text{open}\\ \text{closed}\end{Bmatrix}$ if A and B are $\begin{Bmatrix}\text{open}\\ \text{closed}\end{Bmatrix}$ in X.

A typical $\begin{Bmatrix}\text{open}\\ \text{closed}\end{Bmatrix}$ subset of $A \amalg B$ has the form $C \amalg D$ where C and D are $\begin{Bmatrix}\text{open}\\ \text{closed}\end{Bmatrix}$ in A and B respectively. Therefore, if A and B are $\begin{Bmatrix}\text{open}\\ \text{closed}\end{Bmatrix}$ in X then so are C and D; it follows that $J_0(C \amalg D) = C \cup D$ is $\begin{Bmatrix}\text{open}\\ \text{closed}\end{Bmatrix}$ in X and hence J_0 is an/a $\begin{Bmatrix}\text{open}\\ \text{closed}\end{Bmatrix}$ mapping, so that J is a homeomorphism. $\blacksquare$

Conversely, if A and B are spaces and we are given a homeomorphism $h : C \to D$ where C and D are closed subspaces, then we can form a space X by taking the equivalence relation on $A \amalg B$ generated by $(c, 1) \sim (h(c), 2)$ where $c \in C$.



<u>Claim</u> $\pi \mid A \times \{1\}$, $\pi \mid B \times \{2\}$ are 1-1 closed maps (and automatically continuous)

$A' = \pi[A]$
$B' = \pi[B]$

Hence A' and B' are closed in X, and $A' \cap B'$ corresponds to $C \subseteq A$ and $D \subseteq B$.

To prove the claim, let $E \subseteq A$ and $F \subseteq B$ be closed. Then $\pi[E]$ and $\pi[F]$ are closed $\Longleftrightarrow$ $\pi^{-1}[\pi[E]]$, $\pi^{-1}[\pi[F]]$ are. But

$$\pi^{-1}[\pi[E]] = E \amalg h[E \cap C]$$

closed since $h[E \cap C]$ is closed in D and B because h is a homeo $C \to D$

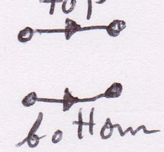
$$\pi^{-1}[\pi[F]] = h^{-1}[F \cap D] \amalg F$$

closed for similar reasons — $h^{-1} : D \to C$ is a homeo

and hence $\pi[E]$, $\pi[F]$ are closed in X.

Similar considerations hold if A and B are open instead of closed.

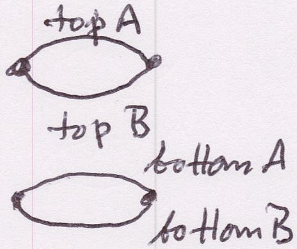
$\boxed{\text{IMPORTANT}}$ Different choices for h may yield topologically inequivalent spaces.



Let $A = B = [0,1] \times \{0,1\}$

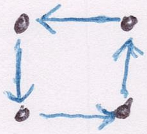
$C = D = \{0,1\} \times \{0,1\}$.

If h = identity, we get



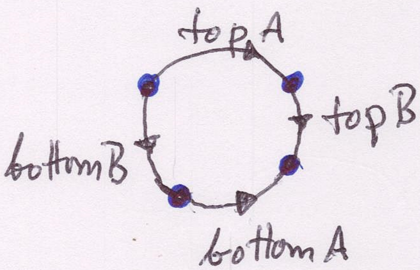
two circles

If h =



, we get

h:
$(1,1) \to (0,1)$
$(0,1) \to (0,0)$
$(0,0) \to (1,0)$
$(1,0) \to (1,1)$



one circle

[Suggestion – Check the latter by cutting and labeling strips of paper!]

Extensions of Theorem II.3.1

Similar results hold if X is described as a union of more than two pieces, say $X = \cup Y_\alpha$ where either of the following is true:

(i) The Y_α are all open in X

(ii) The Y_α are all closed in X <u>and</u> the family of subsets is <u>finite</u>.