Second addendum to III.4

Pages 351–353 of Munkres discuss an application of the Brouwer Fixed Point Theorem to a result about matrices with nonnegative entries.

PERRON–FROBENIUS THEOREM. *If A is an invertible 3×3 matrix over the real numbers with nonnegative entries, it has a positive eigenvalue with an eigenvector whose coordinates are all nonnegative. Furthermore, if the entries of the matrix are all positive, then it has a positive eigenvalue with an eigenvector whose coordinates are all nonnegative.*

The case of matrices with positive entries is discussed as Corollary 55.7, and the case of matrices with nonnegative entries is mentioned as Exercise 3.

At the top of page 352 there is an assertion that *if B denotes the intersection of S^2 with the closed first octant in $\mathbb{R}^3$ (points whose coordinates are all nonnegative), then "it is easy to show that B is homeomorphic to the ball B^2 [$= D^2$], so that the [Brouwer] fixed-point theorem holds for continuous maps of B into itself* (by Corollary III.4.3 in `notes3.4.pdf`). — Statements of the form, "it is easy to show that," are often misleading and should be interpreted as assertions that something is intuitively clear but the proof might be long or inelegant. In the specific example, it might not be difficult to conclude that the two spaces in question seem to be homeomorphic, but writing down a rigorous proof turns out to be more complicated than one might expect. For the sake of completeness we shall sketch a proof here.

One can interpret B geographically as the set of all points which are in the northern hemisphere with longitude between $0°$ and $90°$E. This is a solid spherical triangle whose vertices are the three standard basis vectors $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_1 = (0,1,0)$ and $\mathbf{e}_1 = (0,0,1)$; in Figure 1 on the last page of this document, the set B is shaded in gray. The first step is to flatten B out into the solid triangular region Δ whose boundary triangle has the same three vertices. This triangle lies on the plane defined by the equation $x+y+z = 1$, and we can flatten out the spherical triangle using the mapping

$$F(x,y,z) \quad = \quad \tfrac{1}{x+y+z} \cdot (x,y,z) \ .$$

In order to justify this formula, we need to check that if (x,y,z) lies in the first octant and $x^2+y^2+z^2 = 1$, then $x+y+z > 0$; this implication is true because $x^2+y^2+z^2 = 1$ implies that at least one of the terms x^2, y^2, z^2 is nonzero, which in turn implies that the corresponding coordinate x, y, z is positive. Since the point lies in the first octant, if at least one coordinate is positive then the sum $x+y+z$ of the coordinates is also positive, and therefore the fraction on the right hand side has a nonzero denominator. — One can now check directly that F defines a continuous and 1–1 onto map from B to Δ with inverse given by $G(v) = |v|^{-1} \cdot v$, and since both B and Δ are closed bounded subsets of $\mathbb{R}^3$ it follows that F is a homeomorphism.

The next step is to move and shrink the equilateral triangular set Δ into a similar set T in D^2 whose center is the origin and whose vertices all lie on the second triangle. We shall take the latter to have vertices given by the complex numbers 1, $\omega = \frac{1}{2}(-1+\sqrt{3})$ and $\omega^2 = \frac{1}{2}(-1-\sqrt{3})$ (see the drawing at the end of this file). We claim that an explicit homeomorphism $\Delta \to T$ is given by the formula

$$h(x,y,z) \quad = \quad x\omega \ + \ y\omega^2 \ + \ z \ .$$

There are few things that must be verified. First, we must show that the point in question lies in the solid triangular region of $\mathbb{R}^2$ with vertices 1, ω and ω^2. This is the region bounded by the lines $u = -\frac{1}{2}$ and $1 - u \pm \sqrt{3}\,v = 0$ which join the three pairs of vertices. More precisely, the solid triangular region in $\mathbb{R}^2$ with the given vertices is the set of all points $(u,v) \in \mathbb{R}^2$ such that $-\frac{1}{2} \leq u \leq 1$ and $\sqrt{3}\,|v| \leq 1 - u$. If we write $h(x,y,z) = (u(x,y,z),\ v(x,y,z))$ then we have

$$u \;=\; 1 \;-\; \tfrac{3}{2}\,(x+y)\ , \qquad v = \sqrt{3}\cdot(x-y)$$

and it is a routine exercise to show that u and v satisfy the inequalities defining the solid triangular region in the plane. This system of equations can be solved for x and y uniquely in terms of u and v, and one can then check that if u and v satisfy the defining inequalities then we also have $x, y \geq 0$ and $z = 1 - x - y \geq 0$. Therefore the mapping h is 1–1 onto, and the inverse function is continuous by the usual determinant formulas for solving systems of two linear equations in two unknowns.

Finally, we need to show that T is homeomorphic to the unit disk D^2. By construction, it is contained in the unit disk. We shall define a homeomorphism from D^2 to T using the fact that polar coordinates yield a description ofo D^2 as the quotient space of $[0,1] \times [0,2\pi]$ via the map sending (r,θ) to $(r\cos\theta, r\sin\theta)$; if we take the equivalence relation $\mathcal{E}$ whose equivalence classes are $\{0\} \times [0,2\pi]$, $\{r\} \times \{0,2\pi\}$ for $r > 0$, and (r,θ) for $r > 0$ and $0 < \theta < 2\pi$, then the usual sorts of arguments show that D^2 is the topological quotient space of $[0,1] \times [0,2\pi]/\mathcal{E}$.

The center of the solid triangle T is the origin, which of course is also the center of D^2, and convexity considerations imply that T is contained in D^2 (the set T is the smallest convex subset containing the three vertices, and all three of these vertices lie in the boundary of D^2). Geometrically, a homeomorphism from D^2 to T is given by taking a radial segment in the disk and shrinking it linearly so that the center point is left fixed and the point of intersection with the circle meets the point of intersection with the boundary of the solid triangular region. In the Figure 2 on the next page, this shrinking of the larger segment to the smaller one is indicated by colors (the dark and light pieces of similar colors are radial segments, and the dark pieces are their images under the shrinking map).

The algebraic definition of the mapping $\sigma : D^2 \to T$ has three cases:

$$\sigma[r,\theta] \;=\; \left(\frac{r}{\cos\theta - \sqrt{3}\,\sin\theta},\ \theta\right) \qquad \left(0 \leq \theta \leq \tfrac{2\pi}{3}\right)$$

$$\sigma[r,\theta] \;=\; \left(\frac{r}{2\cos\theta},\ \theta\right) \qquad \left(\tfrac{2\pi}{3} \leq \theta \leq \tfrac{4\pi}{3}\right)$$

$$\sigma[r,\theta] \;=\; \left(\frac{r}{\cos\theta + \sqrt{3}\,\sin\theta},\ \theta\right) \qquad \left(\tfrac{4\pi}{3} \leq \theta \leq 2\pi\right)$$

In each case one needs to check that the denominator is greater than or equal to 1 in order to ensure the formulas are valid (the denominators are never zero) and have the property that $|\sigma[r,\theta]| \leq r$.

To prove that σ defines a homeomorphism, it suffices to note that if λ is any continuous function on $[0,2\pi]$ such that $0 < \lambda(\theta) \leq 1$ then the map $M(v) = \lambda(\theta)v$ is continuous and 1–1, and hence it defines a homeomorphism from D^2 onto its image such that the origin is fixed (the domain is compact and the image is a metric space).

To summarize, we have now shown that the original spherical triangle is homeomorphic to the triangle in the plane of the three unit vectors, the latter is homeomorphic to a standard equilateral triangle inscribed in the unit circle, and the inscribed triangle is homeomorphic to D^2. Therefore we have shown that the original spherical triangle is homeomorphic to D^2, which was our objective.■

GENERALIZATIONS. The Brouwer Fixed Point Theorem generalizes to continuous self-maps of D^n for $n \geq 3$, and the Perron-Frobenius Theorem generalizes to $n \times n$ matrices for $n \geq 4$. Details appear in Sections VII.1 and VII.2 of the following document:

`http://math.ucr.edu/~res/math205B-2012/algtop-notes.pdf`

Drawings for the preceding discussion

The spherical triangle $\boldsymbol{B}$ in the discussion is the shaded portion of the sphere in the first drawing. The vertices of this triangle are the standard unit vectors in coordinate space.

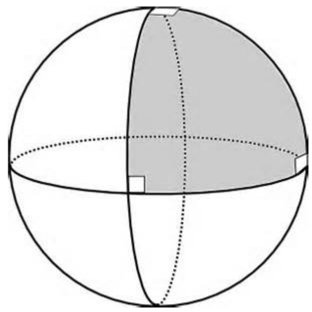

Figure 1

The second drawing depicts the equilateral triangular region $\boldsymbol{T}$ in the coordinate plane. The vertices of this triangle are the complex cube roots of 1, and the discussion on the preceding two pages implies that $\boldsymbol{B}$ is homeomorphic to $\boldsymbol{T}$. As indicated on the preceding pages, there is a homeomorphism from the solid unit disk to $\boldsymbol{T}$ which shrinks radial segments for the disk so that they become segments joining the center of the boundary triangle for $\boldsymbol{T}$ to points on that boundary.

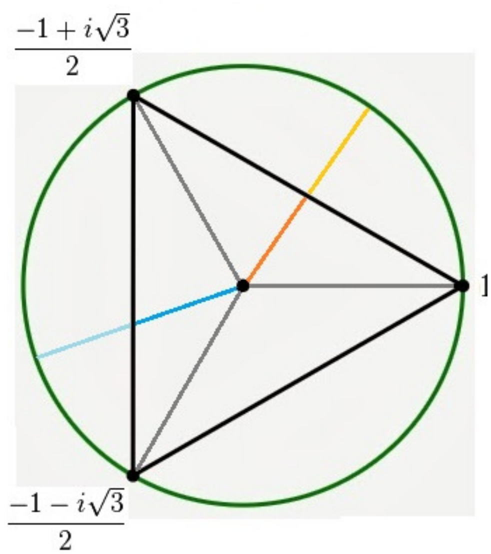

Figure 2