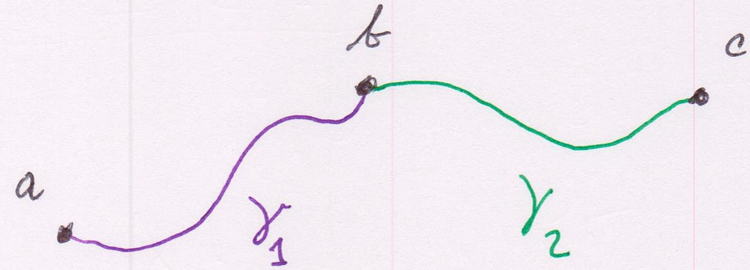
IV.2 Algebraic structure

RECALL Concatenation for curves



γ_1 & γ_2 both defined and continuous on $[0,1]$, with images in a space X.

$$\gamma_1 + \gamma_2 (t) = \begin{cases} \gamma_1(2t) & 0 \le t \le 1/2 \\ \gamma_2(2t-1) & 1/2 \le t \le 1 \end{cases}$$

We get a well defined curve because $\gamma_1(1) = \gamma_2(0) = b$.

The reason for using "+" is that if X is open in $\mathbb{R}^n$, with γ_1 & γ_2 both rectifiable and $P_1, \ldots, P_n : X \longrightarrow \mathbb{R}$ have continuous partials, then

$$\int_{\gamma_1+\gamma_2} \sum P_i \, dx_i = \int_{\gamma_1} \sum P_i \, dx_i + \int_{\gamma_2} \sum P_i \, dx_i .$$

<u>HOWEVER</u> $\gamma_1 + \gamma_2 \neq \gamma_2 + \gamma_1$. For example, if a, b, c are distinct, then $\gamma_2 + \gamma_1$ is not definable even though $\gamma_1 + \gamma_2$ is. Furthermore, $\gamma_1 + \gamma_2 \neq \gamma_2 + \gamma_1$ even when $a = b = c$ in most cases.

<u>Lemma IV.2.1</u> There is a 1-1 correspondence between base point preserving maps $g : (S^1, 1) \longrightarrow (X, x_0)$ and continuous curves $\gamma : [0,1] \longrightarrow (X, x_0)$ such that $\gamma(0) = \gamma(1) = x_0$.

<u>Sketch of proof.</u> $\sigma : [0,1] \xrightarrow[\text{map}]{\text{quotient}} [0,1]/0 \sim 1 \cong S^1$

Then the correspondence is given by

$$\gamma \longleftrightarrow g\sigma . \blacksquare$$

Lemma IV.2.2 Two maps $g_0, g_1 : (S^1, 1) \longrightarrow (X, x_0)$ are base point preservingly homotopic $\Longleftrightarrow$ $g_0\sigma \simeq g_1\sigma$ by a map $H : [0,1]\times[0,1] \longrightarrow X$ which satisfies $H(0,s) = x_0 = H(1,s')$ for all $s, s' \in [0,1]$.

Proof. ($\Longrightarrow$) Let $h : S^1 \times [0,1] \longrightarrow X$ be a basepoint preserving homotopy, and let

$$H(t,s) = h(\sigma(t), s).$$

($\Longleftarrow$) Given h there is a unique map

$$h' : [0,1]\times[0,1] \Big/ \underset{\text{all } s}{(0,s)\sim(1,s)} \longrightarrow X$$

such that h' maps the class of (t,s) to $h(\sigma(t),s)$.

We have already seen that the 1-1 onto cont. map φ is a homeomorphism

$$\begin{tikzcd}
& {[0,1]\times[0,1]} \arrow[dl, "\text{QUOTIENT}"'] \arrow[dr, "\sigma\times 1"] & \\
{[0,1]\times[0,1] \Big/ \underset{\text{all } s}{(0,s)\sim(1,s)}} \arrow[rr, "\varphi"] & & S^1\times[0,1]
\end{tikzcd}$$

So we can take $H = h' \circ \varphi^{-1}$. ∎

<u>Theorem IV. 2. 3</u> Concatenation yields a well-defined binary operation on $\pi_1(X, x_0) = [(S^1, 1), (X, x_0)]$. Furthermore, if $f : (X, x_0) \to (Y, y_0)$ is continuous, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ satisfies $f_*(uv) = f_*(u) \cdot f_*(v)$.

<u>Proof.</u> Given $a, b : (S^1, 1) \to (X, x_0)$, the concatenation $a\sigma + b\sigma$ defines a closed curve in (X, x_0) which we shall call $a + b$.

We need to show

(1) $a \underset{*}{\simeq} a'$ and $b \underset{*}{\simeq} b' \Rightarrow a + b \underset{*}{\simeq} a' + b'$

(2) $f \circ (a + b) = (f \circ a) + (f \circ b)$.

<u>Verification of (2):</u> We have

$$f \circ (a+b) \circ \sigma = f \circ (a\sigma + b\sigma) \underset{\text{THS}}{\overset{\text{VERIFY}}{=\!=}} (f \circ a \circ \sigma) + (f \circ b \circ \sigma)$$

$$\underset{\text{AGAIN}}{\overset{\text{VERIFY}}{=\!=}} \left[(f \circ a) + (f \circ b) \right] \circ \sigma.$$ Since σ is onto,

(2) follows. ■

<u>Note:</u> $\sigma : A \to B$ onto and $g, h : B \to C$ satisfy $g\sigma = h\sigma \Longrightarrow$ $g = h$ since they agree on $\sigma[A] = B$.

<u>Verification of (1):</u> Let $H : a\sigma \simeq a'\sigma$ and $K : b\sigma \simeq b'\sigma$ be homotopies satisfying the condition in IV.2.2, so that both homotopies are constant on $\{0,1\} \times [0,1]$. Define a concatenated homotopy $L : [0,1] \times [0,1] \longrightarrow X$ by

$$L(t,s) = \begin{cases} H(2t, s) & t \leq \frac{1}{2} \\ K(2t-1, s) & t \geq \frac{1}{2}. \end{cases}$$

Then L is a homotopy from $a\sigma + b\sigma$ to $a'\sigma + b'\sigma$ which also satisfies the condition in IV.2.2, and therefore $[a+b] = [a'+b']$ in $\pi_1(X, x_0)$. ■

Here is the ~~m~~ main result.

<u>Theorem IV. 2. 4</u> The binary operation makes $\pi_1(X, x_0)$ into a group.

<u>FACT</u> Every group G is isomorphic to $\pi_1(X_G, x_0)$ for some space X_G.

<u>PROOF</u> Here is what we need to prove:

(1) <u>Associativity</u> $(a+b)+c \underset{*}{\simeq} a+(b+c)$

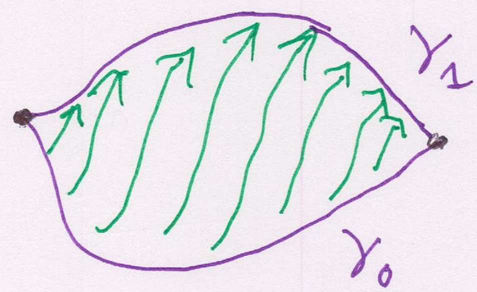
(2) <u>Identity element</u> If K_0 is the constant curve at x_0, then $a + K_0 \underset{*}{\simeq} a \underset{*}{\simeq} K_0 + a$.

(3) <u>Inverses</u> If $(-c)$ is the curve $-c(z) = c(\bar{z} = z^{-1})$ (on S^1), then $(-c)+c \underset{*}{\simeq} K_0 \simeq c+(-c)$.

In fact, all of these hold in a slightly more general setting.

Endpoint preserving homotopy

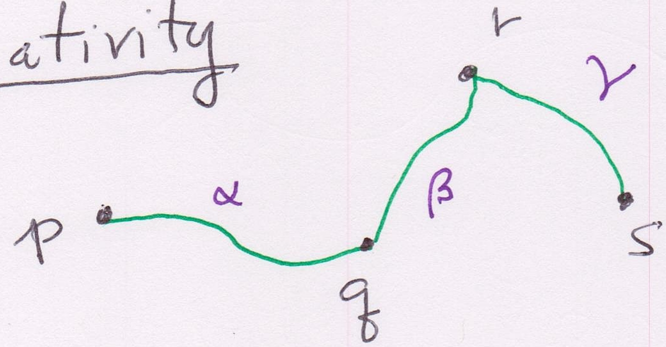
Two curves $\gamma_0, \gamma_1 : [0,1] \longrightarrow X$ are endpoint preservingly homotopic (again write $\gamma_0 \underset{*}{\simeq} \gamma_1$) if there is a homotopy $H : \gamma_0 \simeq \gamma_1$ such that $H(0,s) = \gamma_0(0) = \gamma_1(0)$ and $H(0,s') = \gamma_0(1) = \gamma_1(1)$ for all $s, s' \in [0,1]$



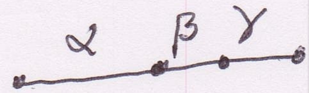
Note If $X \subseteq \mathbb{R}^n$ is convex and γ_0, γ_1 have the same endpoints, then $\gamma_0 \underset{*}{\simeq} \gamma_1$ via the straight line homotopy

$$H(t,s) = (1-s)\gamma_0(t) + s\gamma_1(t).$$

Associativity



$(\alpha+\beta)+\gamma$ vs. $\alpha+(\beta+\gamma)$



Let $h : [0,1] \longrightarrow [0,1]$ be defined as follows:

On $[0, \frac{1}{4}]$ it maps linearly to $[0, \frac{1}{2}]$

On $[\frac{1}{4}, \frac{1}{2}]$ it maps linearly to $[\frac{1}{2}, \frac{3}{4}]$

On $[\frac{1}{2}, 1]$ it maps linearly to $[\frac{3}{4}, 1]$.

Then $(\alpha+\beta)+\gamma = \left[\alpha+(\beta+\gamma)\right]\circ h$.

Note that $h(0)=0$ and $h(1)=1$.

If $K : [0,1]\times[0,1] \longrightarrow [0,1]$ is a straight line homotopy from $id_{[0,1]}$ to h, then we have a homotopy $[\alpha + (\beta+\gamma)] \circ K_t$ from $(\alpha+\beta)+\gamma$ to $\alpha+(\beta+\gamma)$ which is endpoint preserving. ■

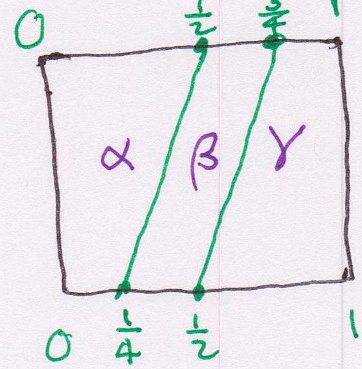


Illustration for K.

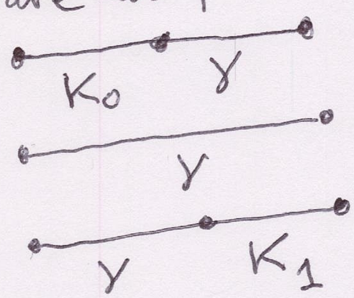
Hence

$(\alpha+\beta)+\gamma \underset{*}{\simeq} \alpha+(\beta+\gamma)$

Concatenation with constants

Compare $K_{\gamma(0)}+\gamma$, γ and $\gamma+K_{\gamma(1)}$;

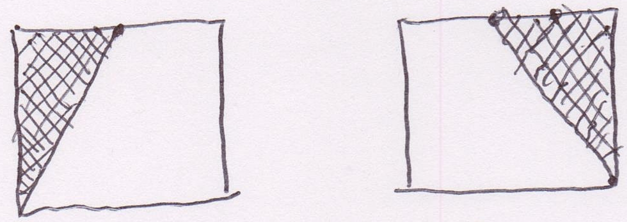
want to show they are endpoint preservingly homotopic.



$K_i = K_{\gamma(i)}$

Proceed as before: $K_0+\gamma = \gamma \circ u$, where $u : [0,1] \longrightarrow [0,1]$ maps $[0,\frac{1}{2}]$ to 0 and $[\frac{1}{2},1]$ to $[0,1]$ linearly. $u + K_1 = \gamma(v)$, where $v : [0,1] \longrightarrow [0,1]$ maps $[0,\frac{1}{2}]$ to $[\frac{1}{2},1]$ linearly and maps $[\frac{1}{2},1]$ to 0. ■

As before, $u, v \underset{*}{\simeq}$ identity, and therefore we have $K_0 + \gamma \underset{*}{\simeq} \gamma \underset{*}{\simeq} \gamma + K_1$. ∎



id $\underset{*}{\simeq} u$ id $\underset{*}{\simeq} v$

The maps are constant on the shaded triangles.

Homotopy inverses

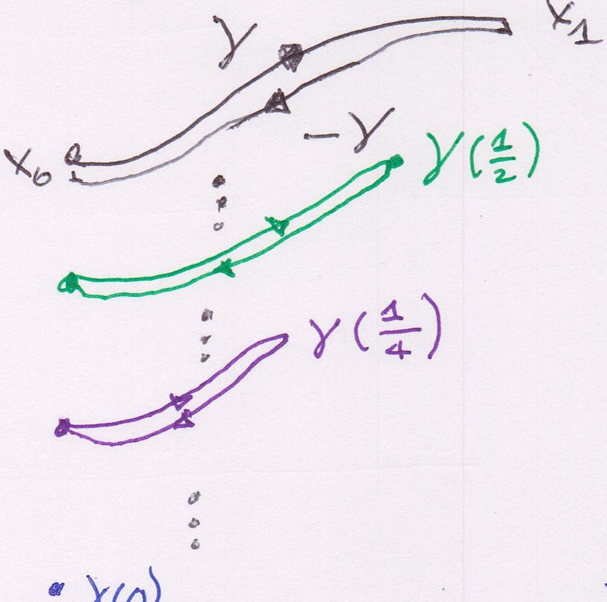
Given a continuous curve $\alpha : [0,1] \to X$, define $-\alpha : [0,1] \to X$ by $-\alpha(t) = \alpha(1-t)$. By construction we have $-(\alpha+\beta) = (-\beta) + (-\alpha)$ and $-(-\alpha) = \alpha$. Therefore we have

$$(-\gamma) + \gamma = (-\gamma) + (-(-\gamma))$$

and if we can show that $\gamma + (-\gamma) \underset{*}{\simeq} K_0$, then we can apply this to $\beta = -\gamma$ and conclude that $(-\gamma) + \gamma \underset{*}{\simeq} K_1$.

Note that if $g : (S^1, 1) \to (X, x_0)$, then $(-g) \circ \sigma = -(g \circ \sigma)$.

The idea behind the construction of a homotopy is fairly simple:



$t = 0$ original curve $\gamma + (-\gamma)$

$t = \frac{1}{2}$ only go to $\gamma(\frac{1}{2})$, then go backwards

$t = \frac{3}{4}$ only go up to $\gamma(\frac{1}{4})$, then go backwards

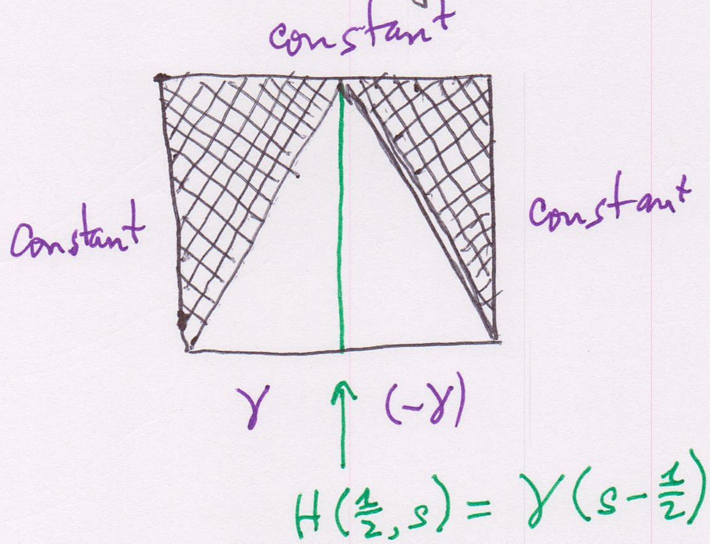
$t = 1$ constant

We can define a homotopy explicitly by

$$H(t,s) = \begin{cases} \gamma(0) & \text{if } t \le \frac{1}{2} \text{ and } s \ge 2t \\ \gamma(t - \frac{s}{2}) & \text{if } t \le \frac{1}{2} \text{ and } s \le 2t \\ H(1-t, s) & \text{if } t \ge \frac{1}{2} \end{cases}$$

The first and second formulas yield the same value if $s = 2t$, and the third formula extends H to $[\frac{1}{2}, 1]$ because $\frac{1}{2} = 1 - \frac{1}{2}$. ∎

Here is a drawing for H:



The homotopy is symmetric with respect to the line $t = \frac{1}{2}$.

Important Addendum to Theorem IV.2: The class of the constant map is the group-theoretic identity element in $\pi_1(X, x_0)$.