Addendum to Section V.2

We shall begin with a more detailed proof of Proposition V.2.3 which corresponds to the presentation in the lectures.

PROPOSITION V.2.3. *If $(X, \mathcal{E})$ is a finite edge-path graph, then X is connected if and only if for each pair of distinct vertices $\mathbf{v}$ and $\mathbf{w}$ there is an* edge-path sequence *$E_1, \cdots, E_n$ such that $\mathbf{v}$ is one vertex of E_1, $\mathbf{w}$ is one vertex of E_n, for each k satisfying $1 < k \leq n$ the edges E_k and E_{k-1} have one vertex in common, and $\mathbf{v}$ and $\mathbf{w}$ are the "other" vertices of E_1 and E_n. Furthermore, X is a union of finitely many components, each of which is a full subgraph.*

Proof. First of all, since every point lies on an edge, it follows that every point lie in the connected component of some vertex. In particular, there are only finitely many connected components, and the number of components is less than or equal to the number of vertices. Define a binary relation on the set of vertices such that $\mathbf{v} \sim \mathbf{w}$ if and only if the two vertices are equal or there is an edge-path sequence as in the statement of the proposition. It is elementary to check that this is an equivalence relation, and that vertices in the same equivalence class determine the same connected component in X; in fact, vertices in the same equivalence class C of this relation will determine the same arc component of X. Note that if E is an edge of the graph and one vertex belongs to C, then so does the other.

Given an equivalence class C, define $\mathcal{H}(C)$, the *hull* of C, to be the union of all edges containing vertices which are in C. By construction, each of these subsets is a full subgraph of $(X, \mathcal{E})$, and each of these subsets is closed in X. Furthermore, given two vertices in C we know that they lie in the same arc component of X, and since every edge in $\mathcal{H}(C)$ has endpoints in C it follows that every edge in $\mathcal{H}(C)$ — and hence $\mathcal{H}(C)$ itself — lies in an arc component of X.

Since every vertex lies in an equivalence class, it follows that X is a union of the finite family of closed subsets $\mathcal{H}(C)$. To conclude the proof, it will suffice to show that these subsets are pairwise disjoint, for then we will have decomposed X into a finite union of closed, arcwise connected, pairwise disjoint subsets. By the finiteness of this collection, the subsets $\mathcal{H}(C)$ are also open (their complements are unions of the remaining closed subsets $\mathcal{H}(C')$), and it follows that the sets $\mathcal{H}(C)$ are the connected and arcwise components of X.

Suppose however that $z \in \mathcal{H}(C) \cap \mathcal{H}(C')$. Since the vertices of $\mathcal{H}(C)$ and $\mathcal{H}(C')$ are disjoint, it follows that z is not a vertex and hence lies on some edge E_0. Now if an edge contains a point of $\mathcal{H}(C)$ or $\mathcal{H}(C')$, then by definition it is entirely contained in the appropriate subset. In particular, $z \in \mathcal{H}(C) \cap \mathcal{H}(C')$ implies $E_0 \subset \mathcal{H}(C) \cap \mathcal{H}(C')$, which in turn implies that the vertices of E_0 also lie in $\mathcal{H}(C) \cap \mathcal{H}(C')$. We have seen that this cannot happen if $C \neq C'$, and therefore it follows that $z \in \mathcal{H}(C) \cap \mathcal{H}(C')$ implies $\mathcal{H}(C) = \mathcal{H}(C')$. This proves that X is a union of the finite family of arcwise connected subgraphs $\mathcal{H}(C)$.

Finally, X is connected if and only if there is only one nonempty subset of the form $\mathcal{H}(C)$, and in this case the construction of the latter implies that every pair of distinct vertices can be joined by an edge-path.■

PROPOSITION 3A. *If $(X, \mathcal{E})$ is a connected graph, then X is arcwise connected.*

Proof. In the final paragraph of the proof, it would be equally valid to say that X is arcwise connected if and only if there is only one nonempty subset of the form $\mathcal{H}(C)$. Therefore if X is arcwise connected it must also be arcwise connected.■