Math 145B
Spring 2017

UPDATED GENERAL INFORMATION — MAY 5, 2017

Review for the midterm examination

The exam will consist of four questions covering topics in the course from connected components through Section 3.2 in the course outline. Here are some problems which may reflect the content of the exam (as usual, there is a lot of subjectivity in any such assessment). Some should be very simple to answer, but others will take more effort. It is also highly worthwhile to go through the material in the file

`aabUpdate03.145B.s15.pdf` (*sic*)

and to revisit the study suggestions that were given for the first quiz in this quarter's course.

(01) Describe the steps in proving that a compact metric space is complete.

(02) Describe the steps in proving the Contraction Lemma.

(03) Describe the steps in verifying that the standard algorithm for computing square roots is valid when $a > 1$: Specifically, the algorithm starts with $x_0 = a$, sets $x_{k+1} = \frac{1}{2}(x_k + (a/x_k))$ for all $k \geq 0$, and has a limit equal to $\sqrt{a}$.

(04) Describe the steps in proving that if X is a metric space with a countable dense subset, then every subset A also has a countable dense subset. [*Hint:* For metric spaces, second countable $\Leftrightarrow$ countable dense subset, and a subset of a second countable space is always second countable.]

(05) Let X be a union of a sequence of subspaces A_n. State whether the following assertion is always true, always false, or sometimes true and sometimes false, giving reasons or examples to support your claims: *If each A_n has a countable dense subset, then X has a countable dense subset.*

(06) Let X be a union of a sequence of subspaces A_n. State whether the following assertion is always true, always false, or sometimes true and sometimes false, giving reasons or examples to support your claims: *If each A_n is compact, then X is compact.*

(07) State whether the following assertion is always true, always false, or sometimes true and sometimes false, giving reasons or examples to support your claims: *If U is an open subset of a connected complete metric space X, then U is also complete.*

(08) State whether the following assertion is always true, always false, or sometimes true and sometimes false, giving reasons or examples to support your claims: *If C is a connected component of a complete metric space X, then C is also complete.*

(09) Let X be a complete metric space, and suppose that C is a connected component of X. Prove that C is complete.

(10) A subspace inclusion $j : A \to X$ is said to be a *retract* if there is a (continuous) one-sided inverse map $r : X \to A$ such that $r \circ j$ is the identity on A. Prove that if A is a retract of

X and $f, g : Y \to A$ are continuous mappings such that $j \circ f$ is homotopic to $j \circ g$, then f is homotopic to g.

(11) Suppose that $A \subset X$ where A is not arcwise connected but X is arcwise connected. Prove that A is not a retract of X.

(12) Let $U \subset \mathbb{R}^2$ be the complement of the points $(\pm 1, 0)$, and let $f : X \to U$ be a continuous mapping such that $|f(x)| \geq 2$ for all X. If $g : X \to U$ is a continuous mapping such that $|g(x) - f(x)| \leq \frac{1}{2}$ for all $x \in X$, prove that g is homotopic to f.

(13) Let K be a convex set, let $p, q \in K$, and consider the constant mappings $C_r : X \to K$ (where $r = p$ or q) whose values everywhere are p and q respectively. Prove that C_p and C_q are homotopic.

(14) Let X be a disjoint union of two copies of $[0, 1]$, and let $\mathcal{R}$ be the equivalence relation generated by identifying the two copies of 0 and likewise for the two copies of 1. Prove that the quotient space $X/\mathcal{R}$ is homeomorphic to the standard unit circle S^1. [*Hint:* The proof of this will require the construction of a continuous mapping from X to the unit circle S^1.]

(15) Prove that A is the set of all values for some Cauchy sequence in a metric space X, then A is a bounded subset of X.

(16) Let X and Y be metric spaces, and take the metric d_p for the product space $X \times Y$, where $p = 1, 2, \infty$. Prove that a sequence $\{z_n = (x_n, y_n)\}$ in $X \times Y$ is a Cauchy sequence if and only if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y respectively.

(17) Why does the preceding result imply that $X \times Y$ is complete (with respect to any of the three product metrics) if and only if X and Y are complete?

Corrections to small (but maybe confusing) typographical errors

In the document `gluing instructions.pdf`, on line 14 of page 3 the space at the end of the line should be $[0, 2] \times \{0\}$; in other words, the last factor should be deleted.

In the document `math145Bnotes3.2.pdf`, on the third line from the bottom of page III.3.2 the space at the end of the line should be $S^1 \times (0, \infty)$. This supersedes the correction indicated in the lectures. Alternatively, the last three lines should be corrected to say that the homeomorphism from $\mathbb{R}^2$ to $S^1 \times \mathbb{R}$ sends z to $(|z|^{-1}z, \ln|z|)$ and the inverse to this map is given by sending (v, t) to $e^t \cdot v$.