Mathematics 145B, Spring 2017, Examination 1

Answer Key

1. [25 *points*] Suppose that (X, d) is a metric space, and suppose that all closed bounded subsets of X are compact. Outline the steps in a proof that X is complete.

SOLUTION

We need to show that every Cauchy sequence in X converges. The first step is to let $\{a_n\}$ be such a sequence and take A to be the set of all its values; *i.e.*, $A = \{a_1, a_2, \cdots\}$. Since $\{a_n\}$ is a Cauchy sequence, A is bounded, and hence by the assumptions its closure $\overline{A}$ is compact.

Next, since $\overline{A}$ is a compact subset containing the original Cauchy sequence, the implication [compact $\Leftrightarrow$ sequentially compact] shows that $\{a_n\}$ has a convergent subsequence, say $\{a_{n_k}\}$. Let b denote the limit of this subsequence.

Finally, one shows that b is the limit of the entire sequence $\{a_n\}$ using the definitions of Cauchy sequence and sequence limit.■

2. [25 *points*] Let X be a topological space which is a union of two connected subsets A and B. Show that X has at most two connected components, and give a pair of examples where the numbers of components are 1 and 2 respectively.

SOLUTION

Every connected subset of X is contained in a connected component of X, so let C and D be the components containing A and B respectively. Then $C \cup D \subset X \subset A \cup B \subset C \cup D$ implies that $X = C \cup D$. Since two connected components are either disjoint or identical, we have that either $C = D$ and X is connected or else $C \neq D$, and X is the union of the two disjoint connected components C and D.

An example where X is connected is given by $[0, 2] = [0, 1] \cup [1, 2]$, and an example where X is not connected is given by $\{0, 1\}$; clearly there are also many other examples of both types.

3. [*25 points*] (a) Suppose that $f : X \to Y$ is a continuous 1–1 onto map of topological spaces and that f is also a quotient map. Prove that the inverse mapping f^{-1} is also continuous. [*Hint:* If $V \subset X$ is open and h is the inverse map, how is the image of V in Y under f related to its inverse image in Y with respect to h?]

SOLUTION

Follow the hint. If f is 1–1 onto and $h = f^{-1}$, then

$$h^{-1}[V] \quad = \quad \left(f^{-1}\right)^{-1}[V] \quad = \quad f[V]$$

(also see Sutherland, Proposition 3.20, p.14). We need to show that if V is open in X, then this set is open in Y.

Since f is a quotient map, a subset $W \subset Y$ is open if and only if $f^{-1}[W]$ is open in X. Therefore, if $W = f[V] = h^{-1}[V]$, then W is open in Y if and only if

$$f^{-1}[W] \quad = \quad f^{-1}\left[f[V]\right] \quad = \quad V$$

is open in X; the right hand equality holds because f is 1–1 and onto. Therefore, if V is open in X it follows that $W = f[V] = h^{-1}[V]$ is open in in Y, and this proves that $h = f^{-1}$ is continuous.■

4. [25 *points*] A subspace $C \subset \mathbb{R}^k$ is said to be a cone if it contains the origin and has the property that $x \in C$ and $t \geq 0$ imply $tx \in C$. Prove that such a subspace C is contractible.

SOLUTION

To simplify notation denote the unit interval $[0, 1]$ by I.

It will suffice to verify that the image of the straight line homotopy $L : C \times I \to \mathbb{R}^k$ defined by $L(x, t) = (1 - t) \cdot x$ is contained in C, for then the associated homotopy $H : C \times I \to C$ defines a homotopy from the identity on C to the constant map sending everything to the origin $\mathbf{0}$.

In other words, we need to verify that $x \in C$ and $t \in I$ imply that $(1-t) \cdot x \in C$. But $t \in I$ implies $1 - t \geq 0$, and therefore $(1 - t) \cdot x \in C$ by the assumption that C is a cone.■