Mathematics 145B, Spring 2017, Examination 2

Answer Key

1. [25 *points*] Let X be a topological space which is a union of two connected subsets A and B. Show that X has at most two connected components, and give a pair of examples where the numbers of components are 1 and 2 respectively.

SOLUTION

Every connected subset of X is contained in a connected component of X, so let C and D be the components containing A and B respectively. Then $C \cup D \subset X \subset A \cup B \subset C \cup D$ implies that $X = C \cup D$. Since two connected components are either disjoint or identical, we have that either $C = D$ and X is connected or else $C \neq D$, and X is the union of the two disjoint connected components C and D.■

An example where X is connected is given by $[0, 2] = [0, 1] \cup [1, 2]$, and an example where X is not connected is given by $\{0, 1\}$; clearly there are also many other examples of both types.■

2. [*15 points*] Let $\mathcal{R}$ be the equivalence relation on the closed interval $[-1, 1]$ whose equivalence classes are the sets P, N, Z of the positive elements, the negative elements, and $\{0\}$ respectively. If X is the quotient space $[-1, 1]/\mathcal{R}$ with the quotient topology, determine all points A in X such that $\{A\}$ is (i) open but not closed, (ii) closed but not open, (iii) both open and closed, (iv) neither open nor closed. — Note that since X has three points and there are four mutually disjoint categories, at least one of these categories will not contain any points at all.

SOLUTION

The singletons $\{P\}$ and $\{N\}$ are both open subsets because their inverse images are open subsets **of the closed interval** $[-1, 1]$; namely, $(0, 1]$ and $[-1, 0)$ respectively. Neither is a closed subset because these half-open intervals are not closed subsets of $[-1, 1]$.

The singleton $\{Z\}$ is a closed subset because its inverse image is the closed subset $\{0\}$. It is not an open subset because $\{0\}$ is not an open subset of $[-1, 1]$.■

To summarize, the first two singletons $\{P\}$ and $\{N\}$ satisfy (i), the zero singleton $\{Z\}$ satisfies (ii), and no singletons satisfy (iii) or (iv).■

3. [10 *points*] Let $f : X \to Y$ be a continuous mapping of topological spaces. State the defining condition for f to be a homotopy equivalence.

SOLUTION

The condition is that there is a continuous mapping $g : Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y.■