EXERCISES FOR MATHEMATICS 145B

SPRING 2015 — Part 2

The remarks at the beginning of Part 1 also apply here. The references denote sections of the texts for the course (Munkres and Crossley).

II. Constructing and deconstructing spaces

II.1 : Disjoint unions

(Crossley, § 5.2)

Additional exercises

1. (*i*) If X and Y are topological spaces, show that $X \amalg Y$ with the disjoint union topology is discrete only if X and Y are discrete.

(*ii*) If X and Y are topological spaces, show that $X \amalg Y$ with the disjoint union topology is Hausdorff only if X and Y are Hausdorff.

(*iii*) Deleted.

(*iv*) If X and Y are topological spaces, show that $X \amalg Y$ with the disjoint union topology is compact only if X and Y are compact.

2. Set theoretically we know that the closed interval $[0,1]$ is in 1–1 correspondence with the disjoint union $[0,\frac{1}{2}) \amalg [\frac{1}{2},1]$. Show that the "identity" map

$$\left[0,\tfrac{1}{2}\right) \amalg \left[\tfrac{1}{2},1\right] \quad \longrightarrow \quad [0,1]$$

is continuous but is not a homeomorphism. [*Hint:* Use Proposition II.1.1 to show the continuity of the identity map. If this map were a homeomorphism, then the domain would be compact and connected.]

3. (*i*) Suppose that we have homeomorphisms of topological spaces given by $f : X \to X'$ and $g : Y \to Y'$. Prove that the map $f \amalg g : X \amalg Y \to X' \amalg Y'$ sending $(x,1)$ to $(f(x),1)$ and $(y,2)$ to $(g(y),2)$ is a homeomorphism.

(*ii*) Suppose that X and Y are topological spaces. Prove that the map $T_{X,Y} : X \amalg Y \to Y \amalg X$ sending $(x,1)$ to $(x,2)$ and $(y,2)$ to $(y,1)$ is a homeomorphism. [*Hint:* Why is $T_{Y,X}$ inverse to $T_{X,Y}$?]

4. Let X_1 and X_2 be topological spaces, and assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for the respective topologies. Prove that a base for the disjoint union topology on $X_1 \amalg X_2$ is given by all sets of the form $U \amalg V$ where $U \in \mathcal{B}_1 \cup \{\emptyset\}$ and $V \in \mathcal{B}_2 \cup \{\emptyset\}$.

5. Suppose that X, Y and Z are topological spaces. Prove that

$$(X \amalg Y) \times Z \qquad \text{and} \qquad (X \times Z) \amalg (Y \times Z)$$

are homeomorphic. [*Hint:* Construct a map from the first space into the second. Use the preceding exercise to show that this map and its inverse send basic open sets in one space to basic open sets in the other space.]

II.2 : Quotient spaces

(Munkres, § 22; Crossley, § 5.1)

Munkres, § 22, pp. 144–145: 4

Additional exercises

1. Suppose that X is a space with the discrete topology and $\mathcal{R}$ is an equivalence relation on X. Prove that the quotient topology on $X/\mathcal{R}$ is discrete.

2. Let $\mathcal{R}$ be an equivalence relation on a space X, and assume that $A \subset X$ is a set which contains points from every equivalence class of $\mathcal{R}$. Let $\mathcal{R}_0$ be the induced equivalence relation on A, and let

$$j : A/\,\mathcal{R}_0 \;\longrightarrow\; X/\,\mathcal{R}$$

be the associated $1-1$ correspondence of equivalence classes. Prove that j is a homeomorphism if there is a continuous map $q : X \to A$ such that $q|A$ is the identity and $u\,\mathcal{R}\,v$ implies $q(u)\,\mathcal{R}_0\,q(v)$ for all $u, v \in X$.

3. (a) Let $\mathbf{0}$ denote the origin in $\mathbb{R}^3$. In $\mathbb{R}^3 - \{\mathbf{0}\}$ define $x\,\mathcal{R}\,y$ if y is a nonzero multiple of x (geometrically, if x and y lie on a line through the origin). Show that $\mathcal{R}$ is an equivalence relation; the quotient space is called the *real projective plane* and denoted by $\mathbb{RP}^2$.

(b) Using the previous exercise, show that $\mathbb{RP}^2$ can also be viewed as the quotient of S^2 modulo the equivalence relation $x \sim y \iff y = \pm\, x$. In particular, this shows that $\mathbb{RP}^2$ is compact. [*Hint:* Let q be the radial compression map that sends v to $|v|^{-1}v$.]

NOTE. One can also prove that $\mathbb{RP}^2$ is Hausdorff, but this property is more difficult to verify. The file `rpn-in-rk.pdf` proves a stronger result; namely, $\mathbb{RP}^2$ is homeomorphic to a subset of $\mathbb{R}^M$ for some positive integer M. A more geometrical proof of the Hausdorff property for $\mathbb{RP}^2$ is contained in the file

`http://math.ucr.edu/~res/progeom/quadrics0.pdf`

and in fact both of these documents state and prove more general results.

4. In the unit disk $D^2 = \{x \in \mathbb{R}^2 |\ |x| \leq 1\}$, consider the equivalence relation generated by the condition $x\,\mathcal{R}'\,y$ if $|x| = |y| = 1$ and $y = -x$. Show that this quotient space is homeomorphic to $\mathbb{RP}^2$.

[*Hints:* Use the description of $\mathbb{RP}^2$ as a quotient space of S^2 from the previous exercise, and let $h : D^2 \to S^2$ be defined by

$$h(x, y) = (x, y, \sqrt{1 - x^2 - y^2}\,).$$

Verify that h preserves equivalence classes and therefore induces a continuous map $\overline{h}$ on quotient spaces. Why is $\overline{h}$ a $1-1$ and onto mapping? Finally, use the facts that $\mathbb{RP}^2$ is Hausdorff and $\overline{h}$ is a closed mapping.]

5. Let $X = [0,1] \times [0,1]$ be the solid closed square, and let $\mathcal{R}$ be the equivalence relation on X whose equivalence classes are $[0,1] \times \{1\}$ and all one point sets of the form $\{(u,v)\}$ where $v < 1$; geometrically, the set quotient is the square with the top edge collapsed to a point. Prove that $X/\mathcal{R}$ is homeomorphic to the solid triangular closed region in $\mathbb{R}^2$ for which the triangle vertices are $(0,0)$, $(1,0)$ and $\left(\frac{1}{2}, 1\right)$. [*Hint:* Draw a picture of the two sets.]

6. Let $X = \mathbb{R} \amalg \mathbb{R}$, and let $\mathcal{E}$ be the equivalence relation whose equivalence classes are the sets $\{t\} \times \{1,2\}$ for $t \neq 0$ and the one point sets $\{(0,1)\}$ and $\{(0,2)\}$. Prove that $X/\mathcal{E}$ is not Hausdorff. [*Hint:* Consider the two equivalence classes which contain only one point, and show that every open neighborhood for one of these points must also contain the other.]

7. Let X and Y be topological spaces, and define an equivalence relation $\mathcal{R}$ on $X \times Y$ by $(x,y) \sim (x',y')$ if and only if $x = x'$. Show that $X \times Y/\mathcal{R}$ is homeomorphic to X.