EXERCISES FOR MATHEMATICS 145B

SPRING 2015 — Part 3

The remarks at the beginning of Part 1 also apply here. The references denote sections of the texts for the course (Munkres and Crossley).

III. Homotopy

III.1 : Basic definitions

(Munkres, § 51; Crossley, § 6.1)

Munkres, § 51, p. 330: 1 – 3

Additional exercises

1. Let X be a topological space, and let P be a topological space consisting of exactly one point (it has a unique topology). Explain why the set of homotopy classes $[P, X]$ is in 1–1 correspondence with the set of arc components of X.

2. Let Y be a nonempty space with the discrete topology (all subsets are open), and let X be a nonempty connected space. Prove that there is a 1–1 correspondence between $[X, Y]$ and Y.

3. (i) Show that if A is a star convex subset of $\mathbb{R}^n$ in the sense of Munkres, Exercise 1, page 334, then the identity map is homotopic to the constant map which sends every point to the "focal point" a_0 (by definition, this is the point such that for each $x \in A$ the closed segment joining x to a_0 lies in A).

(ii) Suppose that $\{A_\alpha\}$ is a nonempty collection of convex subsets in $\mathbb{R}^n$ and that there is a point p in their intersection. Prove that the union $\cup_\alpha A_\alpha$ is a star convex set.

(iii) Let $Y \subset \mathbb{R}^2$ be the union of the $x-$ and $y-$axes. Show that Y is star convex but not convex.

4. Let W, X and Y be topological spaces, and let $u \in [W, X]$ and $v \in [X, Y]$ be homotopy classes of continuous mappings. Prove that there is a well-defined homotopy class $v \circ u \in [W, Y]$ with the following property: If f and g are representatives for the equivalence classes u and v, then $g \circ f$ is a representative for $v \circ u$. [*Hint:* Use Exercise 1 from Munkres.]

5. Let W be a topological space, and let $f : X \to Y$ be continuous.

(i) Using the preceding exercise, show that there is a well defined map of sets $f_* : [W, X] \to [W, Y]$ such that if $v \in [W, X]$ is represented by $g : W \to X$, then $f_*(v)$ is represented by $f \circ g$. Also, explain why f_* is the identity map if $f = \mathrm{id}_X$.

(ii) Suppose we also have a continuous mapping $h : Y \to Z$. Prove that $(h \circ f)_* = h_* \circ f_*$.

(iii) Similarly, show that there is a well defined map of sets $f^* : [Y, W] \to [X, W]$ such that if $v \in [W, Y]$ is represented by $g : W \to Y$, then $f^*(v)$ is represented by $g \circ f$. Also, explain why f^* is the identity map if $f = \mathrm{id}_Y$.

(iv) Suppose we also have a continuous mapping $h : Z \to X$. Prove that $(f \circ h)^* = h^* \circ f^*$.

III.2 : Homotopy equivalence

(Munkres, § 58; Crossley, § 6.2)

Munkres, § 58, pp. 366–367: 1, 3

Additional exercises

1. If X and Y are topological spaces and $f, g : X \to Y$ are homotopic homeomorphisms, prove that their inverses f^{-1} and g^{-1} are also homotopic. [*Caution:* If H is a homotopy from f to g and $t \in [0,1]$, then the maps $h_t : X \to Y$ given by $h_t \leftrightarrow H|X \times \{t\}$ are not necessarily homeomorphisms. Why are the composites $g^{-1} \circ f \circ f^{-1}$ and $g^{-1} \circ g \circ f^{-1}$ homotopic?]

2. Show that two discrete spaces are homotopy equivalent if and only if they have the same cardinalities.

3. Suppose that X and Y are nonempty spaces such that $X \times Y$ is contractible. Prove that both X and Y are contractible. [*Hint:* If $i : X \to X \times Y$ is a slice inclusion sending x to $(y, 0)$ and $p : X \times Y \to X$ is coordinate projection, then $p \circ i = p \circ \mathrm{id}_{X \times Y} \circ i$ is the identity on X. If $X \times Y$ is contractible, then the identity map is homotopic to a constant map. Apply one of the preceding exercises.]

4. Suppose that we are given continuous mappings $f, g : X \to S^n$ such that $f(x) \neq -g(x)$ for all x. Prove that f is homotopic to g. [*Hint:* If $j : S^n \subset \mathbb{R}^{n+1} - \{\mathbf{0}\}$ is the inclusion map, first show that $j \circ f$ and $j \circ g$ are homotopic. If you try to use a straight line homotopy, remember that you need to verify that the origin is not contained in its image.]

5. Suppose that $0 < a \leq 1$ and consider the off center circle in $\mathbb{C} - \{0\}$ defined by $\varphi_a(z) = z + 1$. Prove that if $a < 1$ then φ_a is homotopic to φ_1 in $\mathbb{C} - \{0\}$. [*Hint:* Show that the image of the homotopy $H(z,t) = z + ta$ does not include 0.]

6. Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be homotopy equivalences of topological spaces. Prove that the product map

$$f_1 \times f_2 : X_1 \times X_2 \longrightarrow Y_1 \times Y_2$$

is also a homotopy equivalence. [*Hint:* Recall that a mapping into a product space is continuous if and only if its coordinate projections are continuous, and use this to construct the required homotopies.]

7. (i) Suppose that a topological space X is equal to $A \cup F$ where A and F are closed subsets, and let $B = A \cap F$. Prove that if B is a strong deformation retract of F, then A is a strong deformation retract of X.

(ii) Suppose that a topological space X is a union of two closed subsets $F_1 \cup F_2$, and let $C = F_1 \cap F_2$. Prove that if C is a strong deformation retract of both F_1 and F_2, then C is also a strong deformation retract of X.

8. Show that a space X is contractible if and only if every continuous map $f : X \to Y$, for arbitrary Y, is homotopic to a constant map. Similarly, show X is contractible if and only if every continuous map $f : Y \to X$ is homotopic to a constant map.

9. (i) Show that a homotopy equivalence $f : X \to Y$ induces a 1–1 correspondence between the set of arc components of X and the set of arc components of Y.

(*ii*) For each arc component A of X, show that f restricts to a homotopy equivalence from A of X to an arc component B of Y.

(*iii*) Prove also the corresponding statements with components instead of arc components.

(*iv*) Why does it follow if the components of a space X coincide with its arc components, then the same holds for any space Y homotopy equivalent to X?

III.3 : The circle

(Munkres, §§ 52, 54; Crossley, § 6.3)

Definition. (For the purposes of this course) If $f : S^1 \to S^1$ is a continuous mapping, then the *degree* of f, written $\deg(f)$ is the integer defined as follows: Let $\omega(t) = \exp 2\pi i\, t$, let $t_0 \in \mathbb{R}$ be such that $p(t_0) = f \circ \omega(0)$ — where $p : \mathbb{R} \to S^1$ is the usual map $p(t) = \exp 2\pi i\, t$, take β to be the unique path lifting of $f \circ \omega$ starting at t_0, and set *deg* (f) equal to the unique integer n such that $\beta(1) = t_0 + n$. This integer exists because $p \circ \beta(1) = p \circ \beta(0)$, which means that $\beta(1) - \beta(0)$ is an integer. Since an arbitrary lifting α of $f \circ \omega$ is given by $\alpha(t) = \beta(t) + m$ for some integer m, it follows that $\deg(f)$ does not depend upon the choice of t_0.

Additional exercises

1. Each of the spaces below is either contractible or homotopy equivalent to S^1. For each example, determine which alternative holds. You do not need to give detailed proofs.

(*a*) The solid torus $D^2 \times S^1$.

(*b*) The cylinder $S^1 \times [0, 1]$.

(*c*) The infinite cylinder $S^1 \times \mathbb{R}$.

(*d*) The set of all points $x \in \mathbb{R}^2$ such that $|x| \geq 1$.

(*e*) The set of all points $x \in \mathbb{R}^2$ such that $|x| > 1$.

(*f*) The set of all points $x \in \mathbb{R}^2$ such that $|x| < 1$.

(*g*) The subset of $\mathbb{R}^2$ given by $S^1 \cup \left(\mathbb{R}^+ \times \{0\}\right)$, where $\mathbb{R}^+$ denotes the positive real numbers.

(*h*) The subset of $\mathbb{R}^2$ given by $S^1 \cup \left(\mathbb{R}^+ \times \mathbb{R}\right)$, where $\mathbb{R}^+$ denotes the positive real numbers.

2. The following questions use the notion of degree for a continuous mapping from S^1 to itself.

(*a*) If $f, g : S^1 \to S^1$ are continuous mappings and we take the complex multiplication operation on $S^1 \subset \mathbb{C}$, define $h(z)$ to be the product $h(z) = f(z) \cdot g(z)$. Show that $\deg(h)$ is equal to $\deg(f) + \deg(g)$. [*Hint:* Recall that the winding map $p : \mathbb{R} \to S^1$ has the multiplicative property $p(t_1 + t_2) = p(t_1) \cdot p(t_2)$.]

(*b*) If $f, g : S^1 \to S^1$ are homotopic continuous mappings, then $\deg(f) = \deg(g)$.

(*c*) If $f, g : S^1 \to S^1$ are continuous mappings, define $h(z)$ to be the composite $h(z) = f \circ g(z)$. Show that $\deg(h)$ is equal to $\deg(f) \cdot \deg(g)$. [*Hint:* Why does it suffice to consider the cases where $f(z) = z^p$ and $g(z) = z^q$ where p and q are the respective degrees? For these examples, the identity is true by the laws of exponents.]

3. Find the mistake in the following argument which purports to show that the mappings $f(z) = z$ and $g(z) = z^2$ from S^1 to itself are homotopic: Let $H(z,t) = z^{t+1}$. Then $H(z,0) = f(z)$ and $H(z,1) = g(z)$.

III.4 : The Brouwer Fixed Point Theorem

(Munkres, § 55; Crossley, § 6.4)

Additional exercises

1. A space X is said to have the **Fixed Point Property** if for each continuous mapping $f : X \to X$ there is some $p \in X$ such that $f(p) = p$. By the Brouwer Fixed Point Theorem and its consequences, a space X has the Fixed Point Property if X is homeomorphic to D^2 (and more generally for all D^n, but this is not proved in the course. In contrast, if $X = S^n$ then the antipodal map $T(x) = -x$ has no fixed points, so S^n does not have the Fixed Point Property.

(a) Prove that if X has the Fixed Point Property, then X is connected.

(b) Prove that if X does not have the Fixed Point Property and Y is an arbitrary space, then $X \times Y$ also does not have the Fixed Point Property.

2. Suppose that X and Y are nonempty topological spaces such that $X \times Y$ has the Fixed Point Property. Prove that X and Y have the fixed point property. [*Hint:* If $f : X \to X$ is continuous, consider the map $f \times \mathrm{id}_Y$.]

3. Let $f : S^1 \to S^1$ be a continuous mapping such that $\deg(f) \neq 1$. Prove that f has a fixed point. [*Hint:* Apply Additional Exercise III.2.4 with $g(x) = -x$. What is the degree of the map $h : S^1 \to S^1$ given by $h(z) = -z$?]