2. The Contraction Lemma

The main result is important for both theoretical and practical purposes.

Theorem I.2.1 (Contraction Lemma)

Let X be a complete metric space and $T: X \to X$ a map such that for some α with $0 < \alpha < 1$ we have $d(T(x), T(y)) \leq \alpha \cdot d(x, y)$ for all $x, y \in X$. Then there is a unique $z \in X$ such that $T(z) = z$.

PROOF. The idea is to start with any x, define the sequence $\{T^n(x)\}$ and take z to be the limit of this sequence.

We show $\lim_{n\to\infty} T^n(x)$ exists using the hypothesis on T and the completeness of X.

I.2.2

To do this, we need to show that $\{T^n(x)\}$ is a Cauchy sequence. Consider

Suppose $m > n$

$$d(T^n(x), T^m(x)) \leq d(T^n(x), T^{n+1}(x)) + \dots + d(T^{m-1}(x), T^m(x)) = \sum_{j=0}^{m-n+1} d(T^{n+j}(x), T^{n+j+1}(x))$$

Now $d(T^k(x), T^{k+1}(x)) \leq \alpha^k d(x, T(x))$ by the contraction hypothesis, so the R.H.S is $\leq$

$$\sum_{j=0}^{m-n+1} \alpha^{n+j} d(x, T(x)) = \frac{\alpha^m - \alpha^n}{1-\alpha} d(x, T(x)).$$ (geometric series)

There is some K such that the last term is less than ε if $m, n \geq K$, and therefore $\{T^n(x)\}$ is Cauchy and has a limit z.

By continuity $T(z) = T\left(\lim_{n\to\infty} T^n(x)\right) = \lim_{n\to\infty} T^{n+1}(x) = z$. It remains to show that z is unique.

If $T(z) = z$ and $T(w) = w$, then

$$d(z, w) = d(T(z), T(w)) \leq \alpha \cdot d(z, w)$$

If $z \neq w$, we get some $\Delta = d(z, w) > 0$ so that $\Delta \leq \alpha \cdot \Delta < \Delta$, which is absurd. Hence we can only have one fixed point. ■

The preceding result is fundamental for proving the usual existence and uniqueness results for solutions to ordinary differential equations, but it also has other uses.

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$p(x) = x^3 - x - 1$ is irreducible over the integers and rational numbers. Since $p(1) < 0 < p(2)$, there is some root between 1 and 2. What is it?

There is a cubic formula like the quadratic formula, but it's far less useful for computational purposes.

We shall use the Contraction Lemma.

Translation: Find T so $T(x) = x \iff p(x) = 0$ and the following hold.

T maps some interval $[a, b]$ to itself.

$|T'| \leq \alpha < 1$ on $[a, b]$

(then Mean Value Thm. $\Rightarrow |T(u) - T(v)| \leq \alpha |u - v|$)

Trial & error leads to $T(x) = \sqrt[3]{x+1}$. Then T maps $[1, 2]$ to $[\sqrt[3]{2}, \sqrt[3]{3}] \subseteq [1, 2]$ and $|T'| \leq \frac{1}{3}$.

Hence the root is $\lim_{n \to \infty} T^n(x)$ for $x \in [1, 2]$.

Start with $x = 1$. After 12 iterations one gets answers of the form 1.3247136 to seven decimal places.

Another standard example is the algorithm for computing $\sqrt{a}$ where $a > 1$; namely, take the limit of $T^n(x)$ as $n \to \infty$, where

$$T(x) = \frac{1}{2}\left(x + \frac{a}{x}\right), \qquad a \leq x^2 \leq a^2.$$

Checking the square root algorithm

Let $a > 1$, and consider the function $T(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$. Then $T(x) = x \iff x = \sqrt{a}$.

Claim: T maps $[\sqrt{a}, a]$ to itself, and for some $\alpha < 1$, $|T'(x)| \leq \alpha$ on $[\sqrt{a}, a]$.

If both are true, then $\sqrt{a} = \lim_{n\to\infty} T^n(a)$ by the Contraction Lemma.

Verification of the claim

We have $T'(x) = \frac{1}{2}\left(1 - \frac{a}{x^2}\right)$ and if $x > \sqrt{a}$ this lies in $[0, \frac{1}{2}]$. Hence $x > a$

$\Longrightarrow T(x) - T(\sqrt{a}) \leq \frac{1}{2}(x - \sqrt{a}) < x - \sqrt{a}$

and since $T(\sqrt{a}) = \sqrt{a}$ it follows that $T(x) < x$. Since $T'(y) > 0$ for $y > \sqrt{a}$ it also follows that $T(x) - T(\sqrt{a}) > 0$, which translates into the inequality $T(x) > T(\sqrt{a}) = \sqrt{a}$. ■