III.2 Homotopy equivalence

There is a corresponding notion of <u>homotopy equivalence</u> for topological spaces.

<u>Recall</u> $f: X \xrightarrow{\text{cont}} Y$ is a homeomorphism $\Leftrightarrow$ there is some $g: Y \xrightarrow{\text{cont}} X$ (namely, f^{-1}) such that $g \circ f = id_X$, $f \circ g = id_Y$. Note that g will be unique.

<u>Def.</u> The map $f: X \to Y$ (cont.) is a <u>homotopy equivalence</u> if there is a map $g: Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.

<u>Note</u> g usually will not be unique, but its homotopy class is uniquely determined.

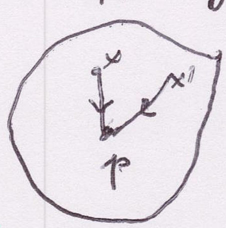
<u>Nonuniqueness of g</u> By the result on page III.15, if $g' \simeq g$ and $f \circ g \simeq id_X$, $g \circ f \simeq id_Y$, then $id_X \simeq f \circ g \simeq f \circ g'$ and $id_Y \simeq g \circ f \simeq g' \circ f$.

<u>Uniqueness of the homotopy class</u> Consider $g' \circ f \circ g$. On one hand we have $g' \circ f \circ g \simeq g' \circ id_Y = g'$ but we also have $g' \circ f \circ g \simeq id_X \circ g = g$. Hence $g' \simeq g$. ∎

EXAMPLES

1. If $X \subseteq \mathbb{R}^n$ is convex and $p \in X$, then $j : \{p\} \subseteq X$ is a homotopy equivalence. — Let $k : X \longrightarrow \{p\}$ be the constant map. Then $k \circ j = id_{\{p\}}$ and $j \circ k \simeq id_X$ by

$$H(x,t) = (1-t)x + tp.$$



straight line homotopy from id to $j \circ k$ = constant map.

2. Each slice inclusion $X \times \{t_0\} \subseteq X \times [0,1]$ is a homotopy equivalence.

$j =$ inclusion. Let $k(x,t) = (x, t_0)$.

Then $k \circ j = id_{X \times \{t_0\}}$, and $j \circ k \simeq id_{X \times [0,1]}$

by $H(x,t;s) = (x, (1-s)t + st_0)$.

vertical homotopy

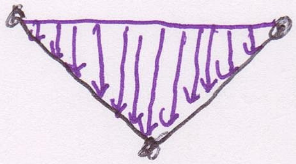
Both of the preceding are special cases of:

<u>Def.</u> Let $j : A \subseteq X$. Then A is a <u>strong deformation retract</u> of X if there is a map $r : X \to A$ such that $r \circ j = id_A$ and $j \circ r \simeq id_X$ by a homotopy which is fixed on A; i.e. $H(x,t) = x$ for all $x \in A$.

Note that $r(x) = H(x,1)$.

Def: $p \in X$. Say X is contractible to p if $\{p\}$ is a strong deformation retract of X.

More examples of deformation retracts

3. 

$-1 \le x \le 1$
$|x| \le y \le 1$

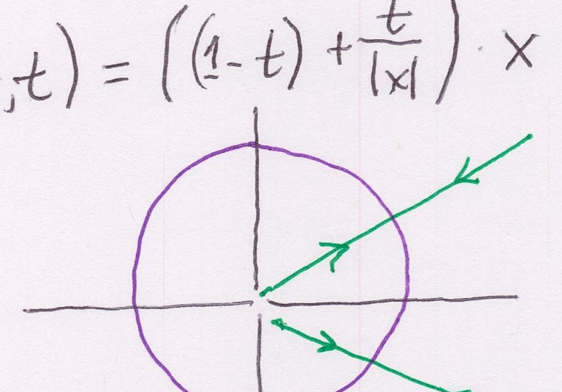
$H(x,y;t) =$
$(x, (1-t)y + t|x|)$.

Another straight line homotopy

4. Let $S^{n-1} \subseteq \mathbb{R}^n$ be all points such that $|v| = 1$.
unit n-sphere.

$S^{n-1} \subseteq \mathbb{R}^n - \{0\}$ is a strong deformation retract.

$H(x,t) = \left((1-t) + \frac{t}{|x|}\right) \cdot x \qquad r(x) = \frac{1}{|x|} x$ unit vector



S^{n-1}

NONEXAMPLES Use

Proposition III.2.1 If $f: X \to Y$ is a homotopy equivalence, then f defines a 1-1 correspondence between the arc components of X and Y.

Proof. Define a map of sets
$f_*: A.C.(X) \longrightarrow A.C.(Y)$ as follows:
Given $x \in X$, let C_x be its arc component.
Since f[Arc conn] is arcwise connected, we have $f[C_x] \subseteq C_{f(x)}$. This is well defined, for $x' \in C_x \Longrightarrow f(x') \in C_{f(x)}$. Furthermore id_{X*} = identity on $A.C.(X)$ and

$$g_* \circ f_* = (g \circ f)_*$$

$$\left(g_* f_*[C_x] = g_*[C_{f(x)}] = C_{g \circ f(x)}\right).$$

Finally, $f \simeq h \Longrightarrow h(x) \in C_{f(x)}$ all x, so

$f_* = h_*$.

If f is a homotopy equivalence, let g be a homotopy inverse. Then we have $g_* f_* = id_{AC(X)}$ and $f_* g_* = id_{AC(Y)}$. Hence f_* is 1-1 onto, and its inverse is g_*. ∎

<u>EXAMPLE</u> $\{0,1\}$ is not homotopy equivalent to an arcwise connected space.

<u>Next question</u> Find examples of arcwise connected spaces that are not homotopy equivalent.