III. 4 The Brouwer Fixed Point Theorem

EXERCISE III. 4. 1. $f : [-1, 1] \to [-1, 1]$ continuous $\Rightarrow$ there is some $x \in [-1, 1]$ such that $f(x) = x$.

SOLUTION. Suppose not. Then we have $g(x) = f(x) - x \neq 0$ for all x.

Claim $g(x) > 0$ or $g(x) < 0$ everywhere, for if $g(x_1) < 0 < g(x_2)$, then there is some y between x_1 and x_2 such that $g(y) = 0$, so that $f(y) = y$.

But $f(-1) \neq -1 \Rightarrow g(-1) > 0$ since $f(-1) > -1$ and similarly $f(1) \neq 1 \Rightarrow f(1) < 1 \Rightarrow g(1) < 0$.

CONTRADICTION. Hence $g(y) - y = 0$ for some y and hence $f(y) = y$ for that value of y. ∎

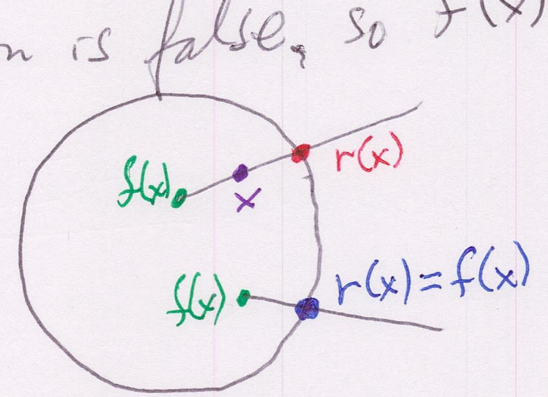
<u>Theorem III.4.2 (Brouwer Fixed Point Theorem)</u> Let $n \geq 1$, and let $f: D^n \to D^n$ be continuous. Then there is some $x \in D^n$ so that $f(x) = x$.

We are only equipped to prove the case $n = 2$ in this course (we have already done the case $n = 1$).

<u>Corollary III.4.3</u> Suppose that X is homeomorphic to D^n $(n \geq 1)$ and $f: X \to X$ is continuous. Then there is some $y \in X$ such that $f(y) = y$.

<u>Proof of Corollary</u> Let $h : D^n \to X$ be a homeomorphism, and let $g : D^n \to D^n$ be $h^{-1} \circ f \circ h$. Then there is some x such that $g(x) = x$. Let $y = h(x)$, so that $x = h^{-1} f(y)$ or equivalently $y = h(x) = f(y)$. ∎

<u>Proof of Theorem III.4.2</u> Suppose the conclusion is false, so $f(x) \neq x$ for all x.



The idea is to construct a continuous map $r : D^n \to S^{n-1}$ as follows: Since $f(x) \neq x$ there is a ray starting at $f(x)$ and passing through x. This ray meets S^{n-1} in a unique point $r(x)$, and $r(x) = x$ if $x \in S^{n-1}$.

The map $x \mapsto r(x)$ is a continuous function of x, and therefore we have a "commutative diagram"

$$\begin{tikzcd} S^{n-1} \arrow[r, "i", "\subseteq"'] \arrow[dr, "\text{identity}"'] & D^n \arrow[d, "r"] \\ & S^{n-1} \end{tikzcd}$$

There are several things that must be checked, and the main issue is the continuity of r. More precisely, we need the following:

(1) There is a unique $t(x) \geq 1$ such that $|r(x)|^2 = |f(x) + t(x - f(x))|^2 = 1$ (on S^{n-1}).

(2) We have $t(x) = 1$ if $x \in S^{n-1}$.

(3) $t(x)$ is continuous in x.

Condition (1) indicates that $t(x)$ is a root of the quadratic equation

$$0 = |f(x)|^2 + 2t\left(f(x)\cdot(x - f(x))\right) + t^2|x - f(x)|^2 - 1$$

and its continuity will follow if we know that there is a unique root with $t(x) \geq 1$. Details are worked out in brouwer.pdf.

At this point we assume $n = 2$

Since every map into D^n is homotopic to a constant, it follows that $i \simeq$ constant and $id(S^1) = r \circ i \simeq r \circ (\text{constant})$ is homotopic to a constant map. But $id(S^1) \not\simeq$ constant, so we have a contradiction. The source of the latter was the assumption that $f(x) \neq x$ for all x, so this must be false and we must have $f(x) = x$ for some x. ∎