IV. Homotopy groups

A central result in Unit III was that $[S^1, S^1] \cong \mathbb{Z}$. We want to have similar group structures on homotopy classes of closed curves into more general spaces. This can be done, but it requires a refinement of the notion of a topological space

IV.1 Pointed spaces

A <u>pointed space</u> or <u>space with base point</u> is a pair (X, x_0) where X is a topological space and $x_0 \in X$.

A continuous mapping (or base point preserving mapping) of pointed spaces $f : (X, x_0) \longrightarrow (Y, y_0)$ is a continuous map $f : X \longrightarrow Y$ such that $f(x_0) = y_0$

<u>Quick observations</u> (1) If f is basepoint preserving from (X, x_0) to (Y, y_0) and g is base point preserving from (Y, y_0) to (Z, z_0), then the composite $g \circ f : X \longrightarrow Z$ is also basepoint preserving.

(2) The identity map on X is basepoint preserving from (X, x_0) to itself (all $x_0 \in X$).

Similarly, if $x_0 \in X \subseteq X'$, then the inclusion of X in X' is a basepoint preserving map from (X, x_0) to (X', x_0).

If $f, g : (X, x_0) \to (Y, y_0)$ are continuous maps, then a basepoint preserving homotopy from f to g is a continuous mapping $H : X \times [0,1] \longrightarrow Y$ such that $H(x,0) = f$, $H(x,1) = g$ and $H \mid \{x_0\} \times [0,1]$ is the constant map with value y_0.

<u>Lemma IV. 1.1</u> If f and g are basepoint preservingly homotopic and g and h are basepoint preservingly homotopic, then so are f and h.

The proof in the unpointed case goes through unchanged.

Likewise, if $\underset{*}{\simeq}$ denotes basepoint preserving homotopy, then $f \underset{*}{\simeq} g \Rightarrow g \underset{*}{\simeq} f$ and $f \underset{*}{\simeq} f$, so we can talk about the base point preserving homotopy classes of maps $[(X, x_0), (Y, y_0)]$.

Furthermore, as before we have $f_0 \underset{*}{\simeq} f_1$ and $g_0 \underset{*}{\simeq} g_1 \Rightarrow g_0 f_0 \underset{*}{\simeq} g_1 f_1$ if $f_0, f_1 : (X, x_0) \longrightarrow (Y, y_0)$ and $g_0, g_1 : (Y, y_0) \longrightarrow (Z, z_0)$.

<u>SIMPLE EXAMPLE</u>

$(S^0, 1)$ where $S^0 = \{\pm 1\} \subseteq \mathbb{R}$.

Then $[(S^0, 1), (X, x_0)]$ is the set of arc components for X and the map sending S^0 to x_0 is a natural base point.

Call this $\pi_0(X, x_0)$.

Given $f : (X_0, x_0) \longrightarrow (Y, y_0)$, the observation in the middle of the preceding page shows that we have a map of pointed sets $f_* : \pi_0(X, x_0) \longrightarrow \pi_0(Y, y_0)$. In more concrete terms, if $x' \in X$ then the equivalence class $[x'] \in \pi_0(X, x_0)$ corresponds to the arc component of x' and $f_*([x'])$ is the arc component of $f(x')$.

<u>Formal properties of f_*</u>

$$(g \circ f)_* = g_* \circ f_* \, , \quad {id_{(X,x_0)}}_* = id \text{ on } \pi_0(X, x_0)$$

The rest of this unit is devoted to analyzing $\pi_1(X, x_0) = [(S^1, 1), (X, x_0)]$. One can also define $\pi_n(X, x_0)$ for $n \geq 2$ (see Crossley), but we shall not study this generalization.