IV, 4 Change of basepoint

Loose ends. If $\alpha, \beta, \alpha', \beta' : [0,1] \to X$ are such that $\alpha(0) = \alpha'(0)$, $\alpha(1) = \alpha'(1) = \beta(0) = \beta'(0)$ and $\beta(1) = \beta'(1)$, and $\alpha \underset{*}{\simeq} \alpha'$, $\beta \underset{*}{\simeq} \beta'$, then $\alpha + \beta \underset{*}{\simeq} \alpha' + \beta'$.

(Just like the proof for closed curves).

Suppose we are given a sequence of curves $\alpha_1, \dots, \alpha_n$ s.t. $\alpha_{i-1}(1) = \alpha_i(0)$. Then one can concatenate these curves in many ways

EXAMPLE

$(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4)$, $\alpha_1 + (\alpha_2 + (\alpha_3 + \alpha_4))$, ...

but all belong to the same endpoint preserving homotopy class, which we shall call

$$[\alpha_1 + \dots + \alpha_n] \text{ or } \left[\sum_{i=1}^{n} \alpha_i\right].$$

(the end result does not depend upon how the parentheses are inserted)

<u>Question</u> If $x_0, x_0' \in X$, how are $\pi_1(X, x_0)$ and $\pi_1(X, x_0')$ related?

<u>Proposition IV.4.1</u> If C is the arc component of x_0, then the inclusion map induces an isomorphism $\pi_1(C, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$

<u>MORAL</u> If x_0 and x_0' lie in separate arc components, one cannot say anything in general.

<u>Proof of Prop. IV.4.1</u> Every closed curve starting and ending at x_0 has an image which is contained in the maximal arcwise connected set C, and likewise for a homotopy between two such curves. $\blacksquare$

In stark contrast, we have

<u>Theorem IV.4.2</u> If X is arcwise connected and $x_0, x_1 \in X$, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.

<u>Proof</u> Let $\gamma : [0,1] \longrightarrow X$ be a curve with $\gamma(0) = x_0$, $\gamma(1) = x_1$. Define $\gamma^* : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ by $\gamma^*([\alpha]) = [(-\gamma) + \alpha + \gamma]$. By the remarks on page 1, this does not depend upon the choice of representative α for a class $u \in \pi_1(X, x_0)$. Furthermore, we have the following identities:

$$(\gamma_1+\gamma_2)^* = \gamma_2^* \gamma_1^* \qquad K_0^* = \text{identity} \qquad \boxed{K_0 = \text{constant}}$$

$$\gamma^*(uv) = \gamma^*(u)\gamma^*(v)$$

If we set $\gamma_1 = \gamma$ and $\gamma_2 = -\gamma$ then the first two identities yield $(-\gamma)^*\gamma^* = \text{identity}$, and likewise we have $\gamma^*(-\gamma)^* = \text{identity}$. Therefore γ^* is 1-1 onto, and by the third identity it is an isomorphism.

VERIFICATIONS

I $(\gamma_1+\gamma_2)^*([\alpha]) = [-(\gamma_1+\gamma_2) + \alpha + (\gamma_1+\gamma_2)] =$
$[(-\gamma_2)+(-\gamma_1) + \alpha + \gamma_1 + \gamma_2] = \gamma_2^*([(-\gamma_1) + \alpha + \gamma_1]) =$
$\gamma_2^*\gamma_1^*([\alpha])$.

II $K_0^*([\alpha]) = [K_0 + \alpha + K_0] = [\alpha + K_0] = [\alpha]$.

III $\gamma^*([\alpha+\beta]) = [(-\gamma) + \alpha + \beta + \gamma] =$

$$[(-\gamma)+\alpha+K_0+\beta+\gamma] =$$
$$[(-\gamma)+\alpha+\gamma+(-\gamma)+\beta+\gamma] =$$
$$[(-\gamma)+\alpha+\gamma]\circ[(-\gamma)+\beta+\gamma] = \gamma^*([\alpha])\gamma^*([\beta]). \blacksquare$$

By construction, γ^* depends only on the end point preserving homotopy class of γ. However, different paths may yield different isomorphisms, even if $x_0 = x_1$ (In this case $K_0^* =$ identity, so we need to give examples of $[\gamma] \in \pi_1(X, x_0)$ where γ^* is not the identity).

<u>Theorem IV.4.3</u> If γ joins x_0 to x_0, then

$$\gamma^*([\alpha]) = [\gamma]^{-1}[\alpha][\gamma].$$

Therefore $\gamma^* =$ identity $\iff [\gamma][\alpha] = [\alpha][\gamma]$ for all $[\alpha] \in \pi_1(X, x_0)$.

The theorem follows directly from the definition of γ^*. ∎

Forgetting the base point

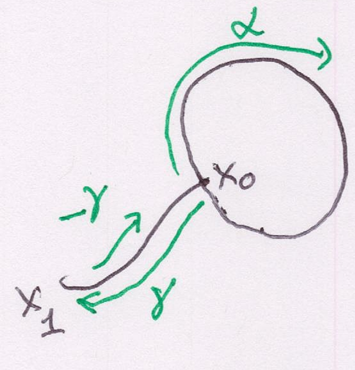
Suppose that X is arcwise connected. What can we say about the forgetful map $\pi_1(X, x_0) \longrightarrow [S^1, X]$?

Theorem IV.4.4 The forgetful map is onto, and two classes $[\alpha], [\beta] \in \pi_1(X, x_0)$ go to the same element of $[S^1, X] \Longleftrightarrow$ there is some $[\gamma] \in \pi_1(X, x_0)$ such that $[\alpha] = \gamma^*([\beta])$.

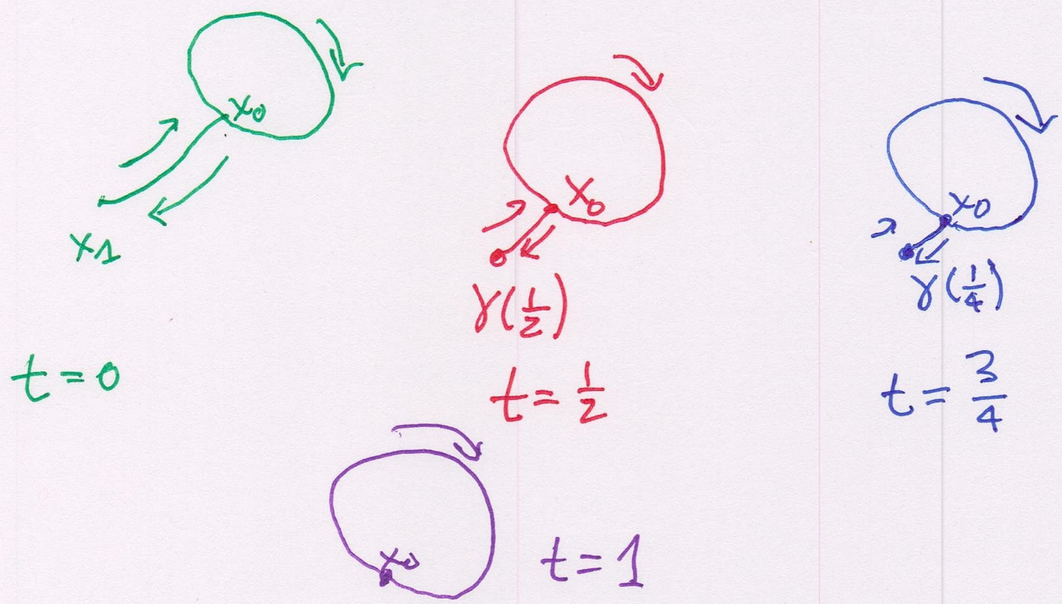
<u>STEP 1</u> If $[\alpha] \in \pi_1(X, x_0)$ and γ is a curve in X joining x_0 and x_1, then $\gamma^*([\alpha])$ and $[\alpha]$ determine the same element of $[S^1, X]$

This implies that $\pi_1(X, x_0) \to [S^1, X]$ is onto (every element of the latter comes from some $\pi_1(X, x_1)$), and it also implies the ($\Leftarrow$) direction of the result.

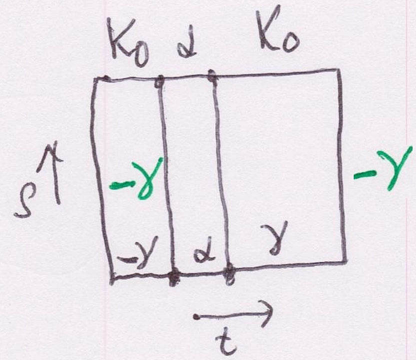
<u>Proof of Step 1</u> We can think of $((-\gamma) + \alpha) + \gamma$ as a balloon on a string:



From this viewpoint, the homotopy is given by "shortening the string" until its length is zero.



Of course we need to write out the homotopy's definition more explicitly, but before doing so we give a picture to illustrate its behavior.



s is the "time" parameter

Formally, $$H(t,s) = \begin{cases} \gamma\big((1-s)(1-4t)\big) & t \le \frac{1}{4} \\ \alpha(t) & \frac{1}{4} \le t \le \frac{1}{2} \\ \gamma\big((1-s)(2t-1)\big) & t \ge \frac{1}{2} \end{cases}$$

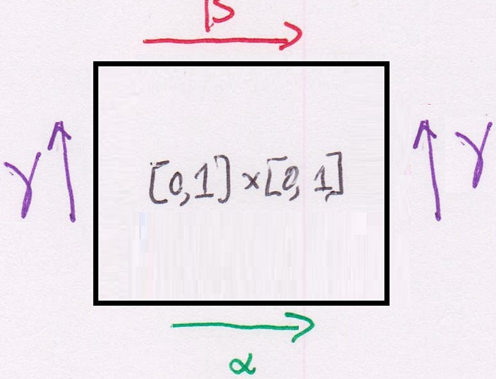
If $t = \frac{1}{4}$ the first and second definitions both yield the same value x_0, and likewise if $t = \frac{1}{2}$ the second and third definitions also yield that value. If $t = 0$ or 1 then

$H(t,s) = \gamma(1-s) = -\gamma(s)$. Hence H passes to a homotopy $S^1 \times [0,1] \longrightarrow X$

STEP 2. If $[\alpha], [\beta] \in \pi_1(X, x_0)$ go to the same element in $[S^1, X]$, then $[\beta] = \gamma^*([\alpha])$ for some α.

Proof of Step 2. Let $h : S^1 \times [0,1] \longrightarrow X$ be a homotopy, and let $\sigma : [0,1] \longrightarrow S^1$ as before.

Now consider $H(t,s) = h(\sigma(t), s)$.



By construction, γ is a closed curve starting & ending at x_0.

The restriction of H to the boundary of the square, parametrized in the counter-clockwise sense, is just

$$(\alpha+\gamma)+((-\beta)+(-\gamma))$$

which represents $[\alpha][\gamma][\beta]^{-1}[\gamma]^{-1}$ in $\pi_1(X,x_0)$. Since the square is convex, this means that the class in question, which comes from the map H, must lie in the image of $\pi_1([0,1]\times[0,1],p) = \{1\}$. Hence $[\alpha][\gamma][\beta]^{-1}[\gamma^{-1}] = 1$, or equivalently $[\alpha]^{-1} = [\gamma][\beta]^{-1}[\gamma]^{-1}$,

or equivalently $[\beta] = \gamma^*([\alpha])$

(work out the algebraic details!). ∎

<u>Corollary III. 4.5</u> If X is arcwise connected and $\pi_1(X, x_0)$ is abelian, then $\pi_1(X, x_0) \cong [S^1, X]$. ∎

(AND CONVERSELY IF X IS ARCWISE CONNECTED).