Solutions & hints for midterm review

This is for the problems in aab Update 05. 145B.s17

(01) The first step is to note that if X is a compact metric space, then every infinite sequence contains a convergent subsequence. The second step is to notice that if a Cauchy sequence has a convergent subsequence, then the limit of the subsequence is in fact the limit of the sequence.

(02) The first step is to consider a sequence of the form $\{T^n x\}$ for an arbitrary $x \in X$. Next, one uses the contraction hypothesis to show that $\{T^n x\}$ is a Cauchy sequence. The third step is to show that if $z = \lim T^n x$, then $Tz = z$ by the continuity of T. Finally, one proves uniqueness using the contraction hypothesis.

(03) The first step is to define $Tx = \frac{1}{2}\left(x + \frac{a}{x}\right)$. Next, one shows that $|T'| \le \alpha < 1$ on $[\sqrt{a}, a]$ for some $\alpha \in (0,1)$. Then one concludes that T is a contraction operator from $[\sqrt{a}, a]$ to itself by the Mean Value Theorem from calculus. The fourth step is to conclude that ~~the unique~~ there is a unique fixed point for T in $[\sqrt{a}, a]$. Finally, one shows that if $Tx = x$, then $x = \sqrt{a}$.

(04) Follow the hint. If X is metric and has a countable dense subset, then X is second countable. Next, use the fact that subspaces of second countable spaces are also second countable, concluding that A is also second countable. Finally, use the fact that if A is second countable, then it has a countable dense subset.

(05) True. If D_n is dense in A_n, then $\cup D_n$ is a countable ~~dense~~ subset in X. To see it is dense in X, let $x \in X$ and let U be an open neighborhood of x in X. Then $x \in A_k$ for some k, and since D_k is dense in A_k there is some $d \in D_k \cap U \subseteq D \cap U$. This means D is dense in X.

~~(06) False. Let $X = \mathbb{R}$, $A_n = [-n, n]$.~~

~~(07) False. Let $X = \mathbb{R}$, $U = (0,1)$.~~

(06) Sometimes true, sometimes false. It's false if $X = \mathbb{R}$ and $A_n = [-n, n]$. However, if $X = A_n$ for some n, then it's true.

(07) Sometimes true, but usually false. It's true if $U = X$. However, if U is a nonempty proper subset, it's false. [next page]

If X is connected, then U as in the prev. sentence cannot be closed, so it is properly contained in $\overline{U}$. Hence there is some limit point $p \in \overline{U}$ with $p \notin U$. This point is the limit of a Cauchy sequence $\{u_n\}$ in U, so there is some Cauchy sequence in the latter with no limit in U.

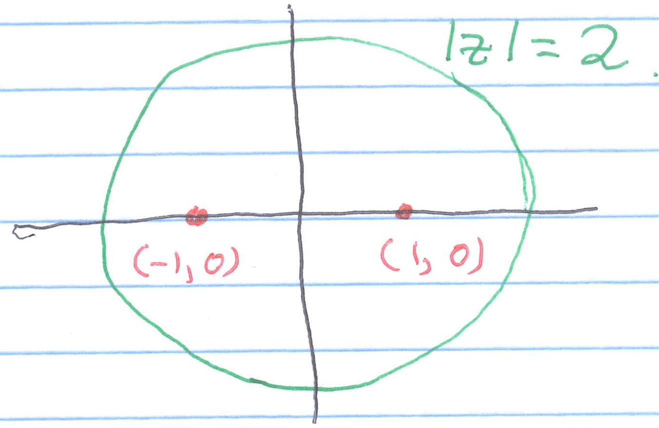
(08) True. A connected component is a closed subset, and closed subsets of complete metric spaces are complete.

(09) See the previous item.

(10) If $j \circ f \simeq j \circ g$, then $r \circ (j \circ f) \simeq r \circ (j \circ g)$. But if $h = f$ or g, then $r \circ (j \circ h) = (r \circ j) \circ h = id_A \circ h = h$. Hence $f \simeq g$.

(11) Let $P = \{p\}$, and let $f(p), g(p)$ belong to different arc components. Then $f \not\simeq g$ but $j \circ f \simeq j \circ g$ by the hypotheses. Apply (10) to conclude A is not a retract of X.

(12) Here is a picture



The simplest attempt to construct a homotopy is to try a straight line homotopy from f to g. and to hope it misses $(\pm 1, 0)$. But if

$$H(t, x) = (1-t) f(x) + t\, g(x) \quad \text{for } 0 \leq t \leq 1$$

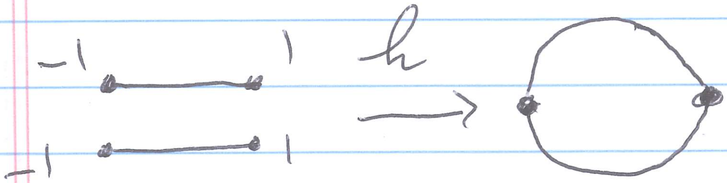
then $|H(t,x)| = |f(x) + t(f(x) - g(x))| \geq$

$$2 - t \cdot \frac{1}{2} \geq \frac{3}{2}, \text{ so that } H(t, x) \neq (\pm 1, 0)$$

for all x and t.

(13) Construct a homotopy of constant maps using the straight line segment from p to q and the convexity of K. Formally $H(x,t) = (1-t)p + tq$, and this lies in K because K is convex.

(14) Here is a drawing, using $[0,1] \cong [-1,1]$



On the first interval, let $h(t) = (t, \sqrt{1-t})$; on the second, let $h(t) = (t, -\sqrt{1-t})$. This passes to continuous 1-1 onto map $\bar{h}: X/R \longrightarrow S^1$. To see $\bar{h}$ is well-defined note that both copies of $\underbrace{\cancel{(-1,0)}}_{-1}$ are sent to $(-1, 0)$ and similarly both copies of $+1$ go to $(+1, 0)$. These yield a continuous 1-1 map $X/R \longrightarrow S^1$. Check it is onto using inverse trigonometric functions, mainly Arccos.

(15) $A = \{a_1, a_2, \ldots\}$. For some M, $n \,\&\, m \geq M \Rightarrow d(a_m, a_n) < 1$, so $n \geq M \Rightarrow d(a_n, a_M) < 1$. Hence $x \in A \Rightarrow d(x, a_M) \leq \max\{d(a_1, a_M), \ldots, d(a_{M-1}, a_M), 1\}$.

$$(16)\quad \left.\begin{aligned} d(x_n, x_m) &\le d(z_n, z_m) \\ d(y_n, y_m) &\le d(z_n, z_m) \end{aligned}\right\} \text{ if } z = (x, y)$$

Hence the coord. projections p_X, p_Y satisfy a Lipschitz condition, so by the quiz problem $\{z_n\}$ Cauchy $\Rightarrow \{x_n\}, \{y_n\}$ Cauchy. Conversely, we have

$$d_p(z_n, z_m) \le d(x_n, x_m) + d(y_n, y_m) \quad p = 1, 2, \infty$$

so if $\{x_n\}$ & $\{y_n\}$ are Cauchy then so is $\{z_n\}$.

(17) ~~Let $\{x_n\}$ be~~ Assume $X, Y \neq \emptyset$.

If $X \times Y$ is complete, X is isometric to each slice $X \times \{y_0\}$, and each of the latter is a closed — hence complete — subset of $X \times Y$. Hence X is complete, and similarly for Y. Conversely, if X and Y are complete and $\{z_n\}$ is a Cauchy sequence in $X \times Y$, then $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y, so they have limits a & b respectively. But then we have $(a, b) = \lim_{n \to \infty} z_n$.