Math 145B
Spring 2015

UPDATED GENERAL INFORMATION — APRIL 23, 2015

References related to the lectures

1. Given two topological spaces X and Y, there was a warning in the lectures that if $\mathcal{R}$ is an equivalence relation on a space X and $\mathcal{R}^*$ is the equivalence relation on $X \times Y$ given by $(x,y)\ \mathcal{R}^*\ (x',y')$ if and only if $x\ \mathcal{R}\ x'$ and $y = y'$, then the natural 1–1 correspondence from $(X \times Y)/\mathcal{R}^*$ to $(X/\mathcal{R}) \times Y$ which sends the class $[(x,y)]$ to $([x],y)$ need not be a homeomorphism. In the terminology of quotient maps, this amounts to saying that if $\pi : X \to X/\mathcal{R}$ is the projection sending a point to its equivalence class, then $\pi \times \mathrm{id} : X \times Y \to (X/\mathcal{R}) \times Y$ is not necessarily a quotient map.

One example can be extracted from Exercise 6 on page 145 of Munkres (if $p \times \mathrm{id}$ were a quotient map then $p \times p = (\mathrm{id} \times p) \circ (p \times \mathrm{id})$ would also be a quotient map), and another is given in Example 7 on pages 143–144 of Munkres. In the first example example the quotient is not Hausdorff, so it may be useful to note that one can also have examples where the quotient space is Hausdorff. A less complicated example of this sort is given on page 111 of the text, *Topology and Groupoids*, by R. Brown. We shall only present the example here. If we let $X = Y = \mathbb{Q}$ and take the equivalence relation $\mathcal{R}$ on $\mathbb{Q}$ defined by $q \equiv q'$ if and only if $q - q' \in \mathbb{Z}$, then the quotient space $\mathbb{Q}/\mathbb{Z}$ is Hausdorff but the standard map $\mathbb{Q} \times \mathbb{Q} \to (\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q}$ is not a quotient map.

2. In the lectures we stated the following result without proof: *If K is a compact subset of $\mathbb{R}^n$ and U is an open subset of $\mathbb{R}^n$, then there are only countably many homotopy classes of continuous mappings from K to U.* — No proof was given because the argument uses a result on uniformly approximating continuous real valued functions on K by polynomials (the Stone-Weierstrass Approximation Theorem), and this result is generally not covered in prerequisite courses. One reference for a proof is Proposition VII.2.2 in the document `fundgp-notes.pdf`, which can be found in the course directory.

More about the first in-class examination

This examination will cover everything through Section III.1 in the course outline (Homotopy: Basic definitions) except for II.3. In particular, this includes 0.6, I.1–I.3 and II.1–II.2 as well as III.1; at various points material from 145A is likely to be needed (see the summary in 0.0–0.5). There will be four main parts, with most if not all divided up into subquestions and opportunities for partial credit. Here are some more specific suggestions. Not everything can appear on an hour examination, but ideally one should know more than the precise contents of such an examination.

(1) For the material on components, all of the proofs for connected components should be understood well enough to explain the steps in the argument, and examples illustrating key points should be known (for example, examples of spaces where components are not open and examples where the connected and arc components are different). Similarities and differences between results about connected and arcwise connected spaces (results about continuous images, products, arc components, etc.) should be understood well enough to quote them or use them in simple situations. A few more specific items are described in the file `aabupdate02`.

(2) The definitions of Cauchy sequences and complete metric spaces should be known, and likewise for the relation between closed and complete subspaces of a complete metric space and fact that compact metric spaces are complete (including the results that an infinite sequence in the latter has a convergent subsequence — however, it is not necessary to understand the proofs well enough to reproduce them).

(3) The steps in the proof of the Contraction Lemma should be understood well enough to explain them informally, and the definition and existence of completions for metric spaces should be known.

(4) The definition and simple properties of disjoint unions should be understood.

(5) The definition and basic properties of the quotient topology should also be understood, including the extent to which certain basic conditions on spaces (compactness, connectedness, Hausdorff, one point subsets closed) carry over to quotient spaces. Given a continuous and onto mapping $f : X \to Y$, conditions for recognizing Y as a quotient space should be known, and some simple examples should be understood (for example, the lateral surface of a cylinder is a quotient of a solid rectangular region).

(6) The basic definition of a homotopy should be understood, and the proof that homotopy is an equivalence relation should be understood well enough to reproduce it. Likewise for straight line homotopies into subspaces of $\mathbb{R}^n$; in the latter case, recall that one must also be able to verify that the image of a straight line homotopy always lies inside the target space.

(7) The compatibility of homotopy with composition of continuous functions should be known well enough to apply it in some simple cases. Here is one example: If either $f : X \to Y$ or $g : Y \to Z$ is homotopic to a constant, show that $g \circ f$ is homotopic to a constant.

Some more specific review problems for Sections 0.6 and I.1 – I.3 were given in `aabupdate02`, and here are a few examples for subsequent sections:

> Let $\mathcal{R}$ be the equivalence relation on S^1 given by $z \equiv w$ if and only if $z = \pm w$. Prove that $S^1/\mathcal{R}$ is homeomorphic to S^1. [*Hint:* Consider the mapping $\psi : S^1 \to S^1$ sending z to z^2.]
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> Let $\mathcal{R}$ be the equivalence relation on $[0, 1]$ whose equivalence classes are $\{0\}$, $\{1\}$ and the open interval $(0, 1)$. List the open subsets of $[0, 1]/\mathcal{R}$.
>
> Let $\mathcal{E}$ be the equivalence relation on $\mathbb{R}$ whose equivalence classes are the rational numbers and the irrational numbers. Prove that the quotient topology on $\mathbb{R}/\mathcal{E}$ is the indiscrete topology.
>
> Let $f, g, h : X \to \mathbb{C} - \{0\}$ be continuous mappings such that $f \simeq g$. If $p(x) = g(x) \cdot h(x)$ and $q(x) = f(x) \cdot h(x)$, prove that $p \simeq q$.

Finally, a review of the homework problems in `exercises1s15.pdf`, `exercises1s15.pdf` and the first section of `exercises3s15.pdf` together with the corresponding `solutions` files is also strongly recommended.