Math 145B
Spring 2015

UPDATED GENERAL INFORMATION — MAY 27, 2015

Study suggestions for the third quiz

This will take place in the discussion section on **Thursday, May 28.** A few suggestions are given below. It would also be worthwhile to review previous `update` files for indications of the sorts of questions which might appear and some specific topics worth reviewing (this applies to preparation for both the first quiz and the first examination!).

Much like the first two quizes, the third one will consist of a few short answer questions. Possibilities include answering questions involving the explicit computations of fundamental groups for spaces like circles or convex sets, the concept of a retract and its implications for fundamental groups, the notion of a fixed point for a continuous mapping, and the concepts of pointed spaces and base point preserving mappings. It is also useful to know examples of subspace inclusions which do not induce 1–1 homomorphisms of fundamental groups and onto mappings of spaces which do not induce onto homomorphisms of fundamental groups (like $f(z) = z^n$ on the circle, where $n \neq 0$).

Coverage will include Sections IV.1 – IV.3 in the course outline and `notes` files. The corresponding sections of Munkres and Crossley are given in `topics145B.pdf`.

Reviewing the homework problems and their solutions in the relevant files is also strongly recommended.

Study suggestions for the second in-class examination

The new material covered on the examination consists of Sections III.2 – III.4, IV.1 – IV.3 and V.2 – V.3 of the `notes` files; as usual, background material from the previous exam is likely to appear in some form although there will be no direct questions on such topics. Material from Section V.4 will appear in an extra credit problem which will be worth 15 points (the required part of the exam will consist of four problems worth a total of 100 points).

Unit III. Many basic definitions and results will appear in some form on the examination. These include homotopy equivalences, homotopy inverses and their uniqueness up to homotopy, strong deformation retracts and the examples in the `notes` files (you should also know how to prove that these are examples), the proof that homotopy equivalent spaces have the same numbers of arc components, the statements of the PLP and CHP, the construction of the isomorphism $[S^1, S^1] \cong \mathbb{Z}$, the concept of a fixed point and the Brouwer Fixed Point Theorem (but it is not necessary to know the proof).

Unit IV. The concepts of pointed spaces and base point preserving maps should be thoroughly understood, and the algebra – up – to – homotopy behind the construction of the fundamental group, and the proof that it is actually a group — should be understood informally. The computations of fundamental group in special cases should be understood well enough to derive them from the triviality of $\pi_1(\{1\}, 1)$ and $\pi_1(S^1, 1) \cong \mathbb{Z}$, and the existence of examples with non-abelian fundamental groups should be known. The material on change of base point will not appear on the examination.

Unit V. As noted previously, Section V.1 will not be covered. Of course the definition of a graph complex $(X, \mathcal{E})$ is central to everything in this unit. Two points of emphasis will be the characterization of connected graphs in terms of edge paths and the use of homology groups to obtain a rough analog of the fundamental group for graph complexes. In particular, this includes the formula for the dimension of the 1-dimensional homology group of a connected graph in term of the sets of vertices and edges, and the computation of the 0-dimensional homology group for a connected graph (but the derivation of this result will not be covered on the examination). There will probably be compuiational questions about the dimensions of 1-dimensional homology for specifically described examples. One strong recommendation is to try and compute this homology for each example of a connected graph which arises in the notes and exercises.

The extra credit problem on Section V.4 will predictably involve the existence of Euler paths. It is worthwhile to understand the proof that such paths exist only if there are 0 or 2 odd vertices, but there is no need to worry about the proof of the converse. However, the ability to analyze examples and give yes/no anwers is likely to be tested; there are numerous examples in the exercises. In cases where an Euler path exists, it is worthwhile to look for an explicit example of a path (these sorts of problems appear often in puzzle magazines and often are a refreshing change of pace).

Practice questions

Here are some further questions to consider in preparation for the quiz or examination:

1. Prove that $S^1 \times S^1$ is not a retract of $D^2 \times D^2$.
2. Let $T : S^1 \times S^1 \to S^1 \times S^1$ be the homeomorphism sending (u, v) to (v, u). Prove that T is not base point preservingly homotopic to the identity. [*Hint:* Show that the induced map of fundamental groups is given by the algebraic map from $\mathbb{Z} \times \mathbb{Z}$ to itself sending (a, b) to (b, a).]
3. Show that the closed annulus in $\mathbb{R}^2$ defined by $\frac{1}{2} \leq |z| \leq 1$ is a strong deformation retract of the sets defined by $|z| \geq \frac{1}{2}$ and $0 < |z| \leq 1$.
4. (a) If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x + 1$, is there a point $p \in \mathbb{R}$ such that f defines a base point preserving map from $(\mathbb{R}, p)$ to itself? If there is no such point, give a proof.
   (b) If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^3 - x$, find all points p such that f defines a base point preserving map from $(\mathbb{R}, p)$ to $(\mathbb{R}, 0)$.
5. Let $(X, \mathcal{E})$ be the connected graph depicted in the file `barycentric.pdf`, and let ω be a linear ordering of its vertices. Compute the dimension of $H_1(X, \mathcal{E}^\omega); \mathbb{F})$ where $\mathbb{F}$ is an arbitrary field.
6. Let X be an arcwise connected topological space containing the distinct points a and b. Let α and γ be continuous curves joining a to b, and let β join b to a. Prove that in $\pi_1(X, a)$ we have the identity $[\alpha + \beta] = [\alpha + (-\gamma)] \cdot [\gamma + \beta]$.
7. This question concerns the Königsberg Bridge Problem: Late in the 19^{th} century an eighth bridge was built as indicated in the first drawing from the file `koenigsberg-sequel.pdf`. This bridge went directly from land mass A to land mass B, going over land mass D with no opportunities to enter or exit. Determine whether the expanded graph has an Euler path.