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Mathematics 145B, Spring 2015, Examination 1

Answer Key

1. [*25 points*] (a) If X is a topological space and C is a connected component of X, explain why C is a closed subset of X.

(b) Give an example of a topological space X and a connected component C such that C is not an open subset of X.

(c) Give an example of a topological space X and an arc component A such that A is not a closed subset of X.

SOLUTION

(a) A component is a maximal connected subset C, and a general result about connected components states that the closure of a connected subset is connected. Therefore the closure $\overline{C}$ of a connected component is also connected. Since $C \subset \overline{C}$ and C is a maximal connected subset, this means that $C = \overline{C}$, which implies that C is a closed subset.■

(b) Two examples are the subset of $\mathbb{R}$ consisting of 0 and all points of the form $\frac{1}{n}$, where n runs through the positive integers, and the set $\mathbb{Q}$ of all rational numbers (in both cases with the subspace topology coming from the usual topology on $\mathbb{R}$.

(c) Let X be the union of the graph of $\sin(1/x)$ for $x > 0$ and the closed line segment $\{0\} \times [-1, 1]$, and let $A \subset X$ be the graph. Then $\overline{A} = X$ and A is an arc component of X.■

2. [*25 points*] Let X be a complete metric space. If $\{x_n\}$ is a sequence of points in X such that for each n we have $d(x_{n+1}, x_{n+2}) \leq \frac{1}{2}\, d(x_n, x_{n+1})$, then an induction argument, the Triangle Inequality and the formula for summing a geometric progression imply that

$$d(x_n, x_{n+k}) \quad \leq \quad \frac{1}{2^{n-1}}\, d(x_0, x_1)$$

for all nonnegative integers n and k. Using this, explain why $\{x_n\}$ has a limit in X.

SOLUTION

Let $\varepsilon > 0$, and choose N so large that $n \geq N$ implies that the right hand side of the displayed inequality is less than ε. Now suppose that $m, n \geq N$. Without loss of generality we might as well assume that $m \geq n$ (otherwise reverse the roles of m and n). Since $m = n + k$ for some nonnegative integer k we can conclude that $d(x_n, x_m) < \varepsilon$ and hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, this sequence has a limit in X.■

3. [*30 points*] For each of the following, determine whether the statement is true or false, and give brief explanations for your answer (one or two sentences should be enough).

(a) If X is a compact topological space and $\mathcal{R}$ is an equivalence relation, then the set of equivalence classes $X/\mathcal{R}$ is compact if it is given the quotient topology.

(b) If X is a Hausdorff topological space and $\mathcal{R}$ is an equivalence relation, then the set of equivalence classes $X/\mathcal{R}$ is Hausdorff if it is given the quotient topology.

(c) There is an equivalence relation $\mathcal{R}$ on the unit interval $[0,1]$ such that $[0,1]/\mathcal{R}$ with the quotient topology is homeomorphic to the circle S^1.

SOLUTION

(a) **TRUE.** The map from X to $X/\mathcal{R}$ which sends a point to its equivalence class is continuous and onto. Since the continuous image of a compact space is compact, it follows that if X is compact then so is $X/\mathcal{R}$.■

(b) **FALSE.** Take $X = [-1,1]$ with the equivalence relation that partitions X into $[-1,0), \{0\} and (0,1]$.■

(c) **TRUE.** If $f(t) = (\cos 2\pi t, \sin 2\pi t)$, then f passes to a continuous map $[0,1]/\mathcal{R} \to S^1$ which is a homeomorphism.■

4. [*20 points*] Let X be a topological space, let Y be a convex subset of $\mathbb{R}^n$ for some n, and let $f, g : X \to Y$ be continuous mappings. Prove that f and g are homotopic.

SOLUTION

Let $H(x,t) = (1-t) \cdot f(x) + t \cdot g(x)$ be the straight line homotopy. Since Y is convex, the image of this map lies in Y and hence defines a homotopy from $X \times [0,1]$ to Y.■