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Mathematics 145B, Spring 2015, Examination 2

Answer Key

1. [*30 points*] (a) Let $f : (S^1, 1) \to (S^1, 1)$ be the mapping $f(z) = z^2$ where squaring refers to complex multiplication on $\mathbb{C} = \mathbb{R}^2$. Using fundamental groups (or some other valid method), prove that f is not base point preservingly homotopic to the identity map.

(b) Prove that S^1 is not a retract of D^2.

SOLUTION

(a) Recall that an isomorphism from $\pi_1(S^1, 1) \cong \mathbb{Z}$ is given by the degree of a mapping, and if $f(z) = z^n$ for some integer n, then the degree of f is equal to n. Taking $n = 1, 2$ we see that the map of the identity map is 1 and the degree of $f(z) = z^2$ is 2. Since the degrees are different, these maps are not homotopic.

(b) Let $e = (1, 0) \in S^1 \subset D^2$, and let $i > S^1 \to D^2$ denote the inclusion mapping. If S^1 is a retract of D^2, then there is a continuous mapping $r : D^2 \to S^1$ such that $r \circ i$ is the identity on S^1. Note that r must be base point preserving because $r(e) = r \circ i(e) = e$. The induced maps of fundamental groups

$$i_* : \pi_1(S^1, e) \longrightarrow \pi_1(D^2, e)\ , \qquad r_* : \pi_1(D^2, e) \longrightarrow \pi_1(S^1, e)$$

then have the property that $r_* \circ i_*$ is the identity on $\pi_1(S^1, e)$. Therefore i_* must be 1–1.

On the other hand, since $\pi_1(S^1, e) \cong \mathbb{Z}$ and $\pi_1(D^2, e)$ is trivial, we know that i_* is definitely not 1–1, so we have a contradiction. The source of the contradiction was our hypothesis about the existence of the mapping r, and therefore no such mapping can exist.■

2. [*25 points*] Let $n \geq 2$. Prove that S^{n-1} is a strong deformation retract of $\mathbb{R}^n - \{\mathbf{0}\}$.

SOLUTION

Let $j : S^{n-1} \to \mathbb{R}^n - \{\mathbf{0}\}$ denote the inclusion mapping, and define a continuous mapping r in the opposite direction by the formula $r(v) = |v|^{-1} \cdot v$. If v is on the unit circle, then $v = r(v) = f \circ j(v)$, so all that remains is to construct a homotopy from $j \circ r$ to the identity such that the homotopy is fixed on $S^{n-1} \times [0,1]$.

The homotopy is given by the straight line homotopy

$$H(v,t) \quad = \quad (1-t)\cdot v \ + \ t \cdot r(v) \quad = \quad \left(1 - t + \frac{t}{|v|}\right) \cdot v \ .$$

The formula yields a point in $\mathbb{R}^n$, and in order to finish the proof we only need to verify that $H(v,t) \in \mathbb{R}^n - \{\mathbf{0}\}$ for all v and t; equivalently, we need to show that $H(v,t)$ is never $\mathbf{0}$. This is apparent if one draws a picture, and to verify this mathematically we need to check that the coefficient of v in the third expression of the display is never zero. Elementary algebra shows that this coefficient is zero if and only if

$$t \quad = \quad \frac{|v|}{|v|-1} \qquad \text{or equivalently} \qquad \frac{1}{t} \quad = \quad 1 \ - \ \frac{1}{|v|} \ .$$

If $0 < t \leq 1$ and $|v| > 0$ this equation is never satisfied because the right hand side is always less than 1 and the left hand side is always at least 1, so $H(v,t)$ is nonzero if $0 < t \leq 1$. In the remaining case where $t = 0$ we have $H(v,0) = v \neq \mathbf{0}$, so it follows that H is never zero on $(\mathbb{R}^n - \{\mathbf{0}\}) \times [0,1]$.■

3. [*20 points*] Determine whether each of the following statements is true or false, and give reasons for your answer.

(a) If (X, x) is a pointed space and $f : X \to Y$ is a continuous mapping of topological spaces, then there is some $y \in Y$ such that f defines a base point preserving continuous mapping from (X, x) to (Y, y).

(b) If (Y, y) is a pointed space and $f : X \to Y$ is a continuous mapping of topological spaces which defines base point preserving continuous mappings from (X, x_0) and (X, x_1) to (Y, y) for $x_0, x_1 \in X$, then $x_0 = x_1$.

SOLUTION

(a) TRUE because we can always take $y = f(x)$.

(b) FALSE because there might be many points in X which map to y. For example, if I denotes the closed unit interval and $f : I \to I$ is defined by $f(x) = x - x^2$, then f defines base point preserving continuous mappings from $(I, 0)$ to $(I, 0)$ and from $(I, 1)$ to $(I, 0)$.■

4. [*25 points*] Let $(X, \mathcal{E})$ be a complete graph for the vertices a, b, c, d; in other words, for each pair of vertices there is a unique edge containing both of them. Explain why X is connected and compute $H_1(X, \mathcal{E}^\omega; \mathbb{F})$ where ω is the alphabetical ordering of the vertices and $\mathbb{F}$ is a field (the answer does not depend upon which field is chosen).

SOLUTION

A graph is connected if and only if every pair of vertices can be joined by an edge path. For this particular example we have a stronger property: Every pair of vertices is joined by an edge. Therefore the space X must be connected.

If E denotes the number of edges in a connected graph $(X, \mathcal{E})$ and V denotes the number of vertices, then we have

$$\dim H_1(X, \mathcal{E}^\omega; \mathbb{F}) \quad = \quad 1 \ + \ E \ - \ V$$

where ω is some linear ordering of the vertices and $\mathbb{F}$ is some field. Note that the right hand side does not depend upon the choices of ω or $\mathbb{F}$.

For the complete graph on 4 vertices we are given that $V = 4$ and $E = 6$, and therefore the dimension is $1 + 6 - 4 = 3$.■

P. S. Cycles representing a basis for H_1 are sketched in the file `exam2s15key-fig1.pdf`.■

5. [*15 points*] For each of the graphs on the next page, determine whether an Euler path exists and give reasons for your answer. If an Euler path exists, there is no need to construct one explicitly for full credit.

SOLUTION

In each case we use the theorem on the existence of Euler paths: Such a path exists if and only if the number of vertices which lie on an odd number of edges is either 0 or 2. In the first case there is a closed Euler path, and in the second case the end points of the path are the two vertices which lie on an odd number of edges.

(a) There are six vertices. Three of them lie on exactly two edges, and the remaining three lie on exactly four edges. Therefore the theorem on the existence of Euler paths implies that such a path exists; in fact, there is a closed Euler path.■

(b) There are nine vertices. Four of them lie on exactly two edges, four of them lie on exactly three edges, and the remaining vertex lies on exactly four edges. Therefore the theorem on the existence of Euler paths implies that no such path exists.■

(c) There are eight vertices. Four of them lie on exactly two edges, two of them lie on exactly four edges, and the remaining two vertices lies on exactly four edges. Therefore the theorem on the existence of Euler paths implies that such a path exists, but there is no closed Euler path.■

P. S. Specific Euler paths for the first and third examples are sketched in the file `exam2s15key-fig2.pdf`; in each case, there are also many other Euler paths.■