Solutions to problems in aab Update09.145B.s17

1. Let $S^1 \subseteq B \subseteq A$ where $A = \{1 \leq x^2+y^2 \leq 2\}$ and $B = \{1 \leq x^2+y^2 < 2\}$, let $j_A, j_B : S^1 \to A, B$ be inclusions and let $r_A, r_B : A, B \to S^1$ be defined by $r(x) = |x|^{-1}x$. Then $r_A \circ j_A = r_B \circ j_B = id_{S^1}$, and $j_A \circ r_A, j_B \circ r_B \simeq id_A, id_B$ by the map

$$h(z,t) = \left[(1-t) + \frac{t}{|z|}\right] \cdot z.$$ We need to check that $h(z,0) = z$, $h(z,1) = |z|^{-1}z$ and if $|z| = 1$ then $h(z,t) = z$ (all t). These are routine.

<u>However</u>, we also need to check that $h(z,t) \in A$ or B if $z \in A$ or B.

Now $1 \leq |z| \leq 2 \overset{0 \leq t \leq 1}{\Longrightarrow} |h(z,t)| = (1-t)|z| + t \geq (1-t)\cdot 1 + t = 1$ and $|h(z,t)| \leq \underset{\text{since } |z| \leq 2}{(1-t)\cdot 2} + t = 2 - t \leq 2$. Furthermore, if $|z| < 2$, then we can strengthen the second inequality to $|h(z,t)| < 2$.

~~Finally, B is not a retract of A. If there were a map $q : A \to B$ such that $q|B$ = identity, and $i : B \to A$ is inclusion, then $iq|B = id|B$, so that iq and $id_A : A \to A$ agree on the dense~~ ~~restricted to~~

This is true because $|z|<2 \Rightarrow$

$h(z,t) = (1-t)|z| + t < (1-t)2 + t =$

$2 - t \leq 2.$

Now proceed to page 2.

FINALLY, B is not a retract of A. Suppose it is, and let $i : B \to A$ denote inclusion. If $q : B \to A$ is a 1-sided inverse, then $q|B = \text{identity}$ and the mappings $i \circ q,\ id_A : A \to A$ agree on B. Now B is dense in A, so this means $i \circ q = id_A$. But if $z \in A - B$ (i.e., $|z| = 2$), then we also have $i(q(z)) \in B$. This contradiction implies that no one-sided continuous inverse q can exist. ■

this pt. is in B

2. Define a map $\varphi : A \sqcup B \longrightarrow E = \{|z| = 2\}$ s.t.

$$\varphi(z) = z \quad \text{if } z \in A = D^2$$

see the drawing on p.5

$$\varphi(x,t) = (1+t)\,x \quad \text{if } x \in B = S^1 \times [0,1].$$

Each of these maps is continuous, so we need to check that points in the same equivalence class with respect to $\sim$ have the same image in E.

Now every nonsingleton equivalence class has the form $\{z, (z,0)\}$ where $|z| = 1$, so we need only check $\varphi_A(z) = z = \varphi_B(z,0)$.

Thus we have a well-defined continuous mapping $f: A \sqcup B/\mathcal{G} \longrightarrow E$.
Since the left side is compact and the right side is Hausdorff, the mapping f will be a homeomorphism if it is 1-1 and onto.

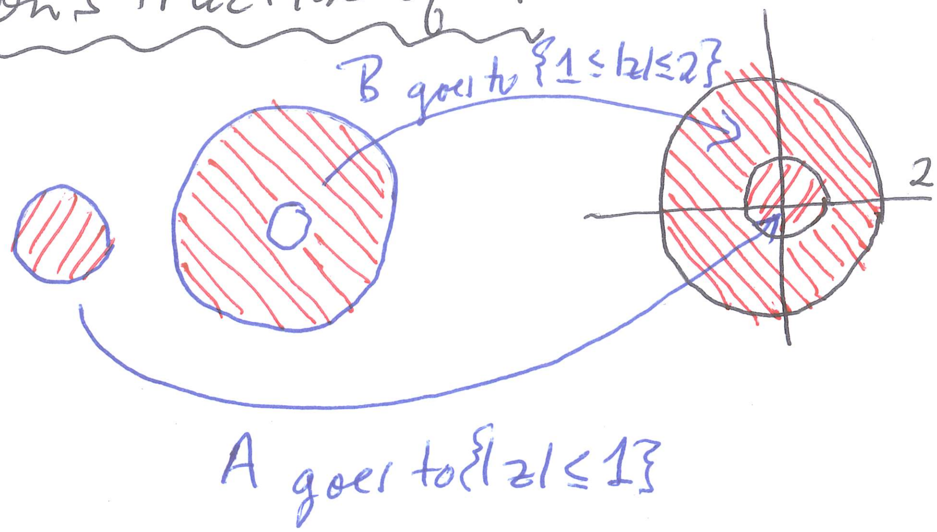
f is onto: If $z \in E$ there are two cases:
$|z| \leq 1$: Then $\varphi(z) = f(z) = z$ by construction.
$|z| \geq 1$: Let $w = |z|^{-1} z$. Then we need to solve $z = (1+t)w$ for ~~t~~ t. Since $|z| = 1+t$, it follows that $t =$ ~~...~~ $|z| - 1$.

f is 1-1: It is enough to show that $\varphi(u) = \varphi(v)$ $\Longrightarrow$ u and v are equivalent under $\mathcal{G}$. This can be done in steps.
First step: Check that both φ_A and φ_B are 1-1, so if $\varphi(u) = \varphi(v)$ then one of $\{u, v\}$ is in A and the other is in B. (These are straightforward!)

Second step: Note that $a \in A$, $b \in B \Rightarrow |\varphi_A(a)| \leq 1 \leq |\varphi_B(b)|$, so $\varphi(a) = \varphi(b) \Longrightarrow |\varphi(a)| = |\varphi(b)| = 1$. If $b = (x, t)$, then then $\varphi(a) = \varphi(b) \Rightarrow a = (1+t)x$, and

$|a| = (1+t) = 1 \Rightarrow t = 0$, and hence $\varphi(a) = \varphi(b) \Rightarrow a = x$, so that $\varphi(a) = \varphi(x,t)$ $\Rightarrow a = x$ and $t = 0$, which means that a and b lie in the same $\mathcal{G}$-equivalence class. Therefore the mapping f is 1-1. $\blacksquare$

Construction of φ



$\varphi[A] \cap \varphi[B]$ is the circle $\{|z| = 1\}$.

Under φ, the unit circle in A is glued to the unit circle in B.