Review of material from prerequisite courses

This is a selective review of some items from 144 and 145A, and to some extent it anticipates how they will be used in 145B. References for sections 0.1 – 0.5 are given on the list of topics for this course, and Chapter 1 of Munkres contains everything about set theory and the real numbers that will be needed.

The first three units of the directory file `gentop-notes.pdf` also contain more detailed discussions of the material summarized below.

0.0 : Set theory

We shall frequently work with equivalence relations in this course, and often we shall describe them in terms of their equivalence classes, which are pairwise disjoint subsets whose union is the entire set under consideration.

The notation for ordered pairs in Munkres is nonstandard, and we shall use the more standard form (x, y) instead of $x \times y$. This notation clashes with the notation for open intervals in the real line, but usually it is clear from the context whether we are referring to an ordered pair or an open interval.

Our concept of function f from A to B (written $f : A \to B$) will consist of three items. One is the set A on which the function is defined (the domain), another is the graph of f viewed as a subset of $A \times B$, and the third is a set B (the co-domain, usually written codomain) in which the function takes values. Often the third item is missing from the definition, but it will be crucial for the purposes of this course, and it corresponds to the need to define the value set for a function in certain computer languages (for example, some functions may be integer valued and some may be real valued). For example, the identity function on the integers $\mathbb{Z}$ and the inclusion function $j : \mathbb{Z} \to \mathbb{R}$ (corresponding to $\mathbb{Z} \subset \mathbb{R}$) both have the same domain and their graphs are the same set, but their codomains differ.

Also, when discussing images and inverse images of $A \subset X$ and $B \subset Y$ of a function $f : X \to Y$ we shall generally use square brackets $f[A]$ and $f^{-1}[B]$ to denote these subsets in order to stress that these are not values of the function but subsets of the domain and codomain respectively.

At various points we shall used basic facts about transfinite cardinal numbers, but we shall not really need to worry about topics like the Axiom of Choice. In some arguments we might simply say, "Choose some point $x \in X$," and all we need to know is that it is somehow possible to do so, even if no specific construction is involved.

0.1 : Metric and topological spaces

Topology is largely devoted to various sorts of objects constructed from the real number system $\mathbb{R}$, so it is absolutely necessary to understand the standard characterization of $\mathbb{R}$ as a complete ordered field (this requires the concepts of least upper and greatest lower bounds).

Most of the specific topological spaces that we consider in this course are *metrizable* (pronounced "metRIZEable"), which come from metric spaces, so it is important to understand how metric spaces fit into the theory of topological spaces. Recall that the topology of open sets on a metric space is determined by the open neighborhoods with specified radii; we shall use $N_r(x)$ to denote the neighborhood of radius r centered at x. The family of such open neighborhoods is a special case of a base for a topology, which is a family $\mathcal{B}$ such that every open subset is a union of subsets in $\mathcal{B}$.

It is worth noting that different metrics on the same set sometimes define the same topology. For example, this is true for $\mathbb{R}^2$ if we take the usual Pythagorean metric or the turtle/taxicab metric, for which the distance between (x_1, y_1) and (x_2, y_2) is equal to $|x_1 - x_2| + |y_1 - y_2|$. A third metric which also defines the same topology is given by taking the maximum of $|x_1 - x_2|$ and $|y_1 - y_2|$.

Topological structures can also be described using closed sets, which are the relative complements of open sets. For metric spaces, the term "closed" can be interpreted as the property of being "closed under taking limits of convergent infinite sequences." However, this does not work for general topological spaces; instead, one has a concept of limit point, and a subset turns out to be closed if and only if it is closed under taking limit points. One important concept related to closed sets is the notion of a closure for an arbitrary subset. A key result states that the closure of a set is the union of a set together with its limit points, and the closure is the unique smallest closed subset with contains the original set. There is also a complementary notion of interior for an arbitrary subset, and there is a complementary result which states that the interior is the unique largest open subset contained in the original set. Finally, there is a hybrid notion of a boundary or frontier for a subset; given a subset $A \subset X$, this is the set of points which are limit points for both A and the relative complement $X - A$.

Finally, if A is a subset of a metric or topological space X, then there is a standard way of making A into a metric or topological space, and this is called the subspace metric or the subspace topology.

0.2 : Continuity

Metric spaces form a natural setting for the notions of continuity and uniform continuity which arise in calculus. A fundamental theorem states that a function on metric spaces is continuous if and only if the inverse images of open subsets in the codomain are open subsets in the domain (equivalently, if and only if the inverse images of closed subsets in the codomain are closed subsets in the domain), and one can use these to define continuity for arbitrary topological spaces. It is important to be able to remember which functions from calculus are continuous and which are not.

Two general facts about continuous functions with far-reaching implications are that identity mappings are always continuous and a composite of two continuous functions is continuous. Given a subspace $A \subset X$, the inclusion map $i : A \to X$, which sends $a \in A$ to itself, is also continuous. Given a continuous mapping $f : X \to Y$, the composite $f \circ i : A \to Y$ is called the restriction of f to A and is denoted by $f|A$. The subspace topology also has the following basic property:

FACT. *Let $f : X \to Y$ be a continuous mapping, let $B \subset Y$ be such that $f[X] \subset B$, and let $j : B \to Y$ be the inclusion map where B has the subspace topology. If $f_0 : X \to B$ is defined by $f_0(x) = f(x) \in B$, then $f = j \circ f_0$ is continuous if and only if f_0 is continuous.*

See Crossley, Proposition 3.35 (p. 28) for more on this result.

If we are given a family of subsets A_α such that $\cup_\alpha \ A_\alpha = X$ and continuous mappings $f_\alpha : A_\alpha \to X$, one frequently arising question is whether there is a continuous function on all of X whose restriction to each A_α is given by f_α. One obvious condition is that the function must be well defined; we must have $f_\alpha|A_\alpha \cap A_\beta = f_\beta|A_\alpha \cap A_\beta$ for all α and β. Simple examples show that the functions f_α need not piece together to form a continuous function on X even if this compatibility condition is satisfied, but one does obtain a continuous function if either of the following is true:

(1) The subsets A_α are all open.

(2) The subsets A_α are closed and there are only finitely many of them.

Finally, there are two other types of mappings for topological spaces that are often important. We say that $f : X \to Y$ is open if images of open subsets are open, and f is closed if images of closed subsets are closed. Given f, it is meaningful to ask whether f is continuous, open or closed. There are eight possible combinations of yes/no answers to these questions, and each possibility can be realized for many different examples.

0.3 : Homeomorphisms

Recall that a mapping of $f : X \to Y$ of topological spaces is a homeomorphism if f is 1–1 onto and both f and its inverse function are continuous. Also, recall that if f is continuous and 1–1 onto then f^{-1} is not necessarily continuous.

Two alternate characterizations of homeomorphisms are that they are 1–1 maps which are both continuous and open, or which are both continuous and closed.

It is extremely useful to think of homeomorphisms between spaces geometrically. For example, a homeomorphism between two subspaces in $\mathbb{R}^n$ can be viewed as a transformation which may warp distances and angles, but does so without tearing or cutting an object. The course directory file `homeomorphisms.pdf` contains more information about this view.

If $f : X \to Y$ is a homeomorphism and $f[A] = B$, then f induces homeomorphisms from A to B and from $X - A$ to $Y - B$. This yields a useful method for showing that two spaces are not homeomorphic. For example, if A is a set with n points, we can conclude that X is not homeomorpic to Y if there is some subset $A \subset X$ with n points such that $X - A$ is not homeomorphic to any subset of Y having the form $Y - B$, where B runs through all subsets which contain exactly n points.

0.4 : Topological properties

There are two families of properties which are particularly important. One involves topological spaces which satisfy versions of results which are true for metric spaces. For example, a $\mathbf{T_1}$ space is a topological space in which all one point subsets are closed, and this includes all metric spaces because one can prove that each set $X - \{x\}$ is open in X if X is a metric space. A stronger and very useful property is the Hausdorff Separation Property, which states that every pair of distinct points has a pair of disjoint open neighborhoods; once again this always holds for metric spaces.

There are numerous reasons why Hausdorff spaces are particularly useful. One is the following property: *If $f, g : X \to Y$ are continuous mappings and Y is Hausdorff, then the set of all points E where $f = g$ is a closed subset of X.*

Another class of topological properties involves abstract versions of facts about closed intervals in the real line which play crucial roles in calculus. These include connectedness, which is the foundation upon which one proves the Intermediate Value Property for continuous functions on intervals (which may be closed, open or half open), and compactness, which is the foundation upon which proves that a continuous function on a closed interval (and more generally a closed bounded subset of $\mathbb{R}^n$) attains a maximum value and a minimum value. These results have generalizations to the continuous images of connected spaces and compact spaces.

The connected subsets of $\mathbb{R}$ are the intervals, but if $n \geq 2$ there is a wide assortment of connected subsets. There are basic results which allow one to show that many naturally occurring subsets of $\mathbb{R}^n$ are either connected or not connected. Another basic result states that a subset of $\mathbb{R}^n$ is compact if and only if it is close and bounded.

Finally, there is a basic result on recognizing homeomorphisms of compact Hausdorff spaces. If a 1–1 onto mapping $f : X \to Y$ from a compact space to a Hausdorff space is continuous, then it is a homeomorphism. Often this is useful for recognizing that certain constructions yield Hausdorff spaces.

0.5 : Products

Given two topological spaces X and Y, there is an elementary way of constructing a topology on $X \times Y$ using "rectangular" open sets which are products of open sets in X and Y. If X and Y are metric spaces, then one can define a metric (in fact, several metrics) on $X \times Y$ which yield this topology.

Product topologies are important for two reasons.

(1) Continuous mappings from $X \times Y$ into a space Z are the basis for talking about continuous functions of two independent variables.

(2) Continuous mappings from a space Z into $X \times Y$ are the basis for talking about parametric equations of curves.

As in the case of sets, one can talk more generally about arbitrary finite products of a sequence of spaces $X_1 \ \cdots \ , X_n$ by means of the recursive definition

$$X_1 \ \cdots \ , X_n \ = \ \big(X_1 \ \cdots \ , X_{n-1}\big) \ \times \ X_n \ .$$

It is also possible to define products for infinite indexed families of spaces, but we shall not need such objects in this course.

One fundamental property of products is their behavior with respect to continuous mappings:

FACT. *Let X and Y be spaces, and let $p_X : X \times Y \to X$, $p_Y : X \times Y \to Y$ be the maps sending (x, y) to x and y respectively. If Z is a topological space and $f : Z \to X \times Y$ is a mapping of sets, then f is continuous if and only if $p_X \circ f$ and $p_Y \circ f$ are continuous.*

See Crossly, Theorem 5.42 (p. 73) for details.

The product construction is compatible with the topological properties in the prededing section. For example, if X and Y are Hausdorff, then so is $X \times Y$. Also, if X and Y are connected, then so is $X \times Y$, and if X and Y are compact, then so is $X \times Y$.