Solutions to Chapter 17 exercises

17.1 Let (x_n) be a Cauchy sequence in a discrete metric space (X, d). There exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < 1$ whenever $m > n \geqslant N$, so in fact $d(x_m, x_n) = 0$ whenever $m > n \geqslant N$, since the metric is discrete. This says that the sequence is 'eventually constant', namely $x_n = x_N$ for all $n \geqslant N$. So (x_n) converges, and X is complete.

17.2 We use the criterion given in Propositions 17.6 and 17.7: a subspace of a complete metric space X is complete iff it is closed in X. We know also that $\mathbb{R}$ and $\mathbb{R}^2$ are complete.

(i) This is complete since it is closed in $\mathbb{R}$ - its complement is the union of open intervals $(-\infty, 0) \cup (1, \infty) \cup \bigcup_{n\in\mathbb{N}} (1/(n+1), 1/n)$ (see Example 6.2 (a) (iv).)

(ii) This is not complete since it is not closed in $\mathbb{R}$ - its closure is $[0, 1]$.

(iii) This is complete since it is closed in $\mathbb{R}^2$ - it is the intersection of two closed sets, namely $\{(x, y) \in \mathbb{R}^2 : x \geqslant 0\}$ and $\{(x, y) \in \mathbb{R}^2 : xy \geqslant 1\}$.

17.3 (a) Suppose that A and B are complete subspaces of a metric space X, and let (x_n) be any Cauchy sequence in $A \cup B$. Then either (x_n) has a subsequence which lies entirely in A, or it has a subsequence in B (or both). Such a subsequence is Cauchy since (x_n) is, so it converges by completeness of A (or of B). But if a Cauchy sequence has a convergent subsequence, the whole sequence converges by Lemma 17.10. So $A \cup B$ is complete.

(b) Suppose for each i in some indexing set I that A_i is a complete subspace of a metric space X, and let (x_n) be a Cauchy sequence in $\bigcap_{i\in I} A_i$, assuming this intersection is non-empty. Then for any particular $i_0 \in I$, the sequence (x_n) is a Cauchy sequence in A_{i_0} hence it converges. (If the intersection is empty then it is vacuously complete, since there are no Cauchy sequences for which to check convergence.)

17.4 As in Definition 6.33, let h, k be positive constants such that $hd'(x, y) \leqslant d(x, y) \leqslant kd'(x, y)$ for all $x, y \in X$, and suppose that (X, d) is complete. Let (x_n) be a Cauchy sequence in (X, d'). We prove that (x_n) is a Cauchy sequence in (X, d). For let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $d'(x_m\, x_n) < \varepsilon/k$ for all $m, n \in \mathbb{N}$ with $m > n \geqslant N$. Then $d(x_m, x_n) < \varepsilon$ for all $m, n \in \mathbb{N}$ with $m > n \geqslant N$. Hence (x_n) is Cauchy in (X, d), and by completeness it converges in (X, d), say to $x \in X$. We prove that (x_n) converges to x also in (X, d'), so the latter is compelete. For let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $d(x_n, x) < h\varepsilon$ for all $n \geqslant N$. Hence $d'(x_n, x) < \varepsilon$ for all $n \geqslant N$.

17.5 Let us label these metrics d_a, d_b, d_c.

(a) Then $(\mathbb{R}, d_a)$ is complete. For let (x_n) be a d_a-Cauchy sequence. Then (x_n^3) is a Cauchy sequence in the ordinary sense, so it converges in the usual sense, say to $y \in \mathbb{R}$. Let $x = y^{1/3}$. Then (x_n) converges to x in the metric d_a. For $d_a(x_n, x) = |x_n^3 - x^3| = |x_n^3 - y| \to 0$ as $n \to \infty$.

(b) This is not complete. For consider the sequence $(-n)$. Then $e^{-n} \to 0$ as $n \to \infty$, so (e^{-n}) is Cauchy in the usual sense, which says that the sequence $(-n)$ is d_b-Cauchy. But suppose that $(-n)$ converged to a real number a in $(\mathbb{R}, d_b)$. Then $d_b(-n, a) \to 0$ as $n \to \infty$, or equivalently $|e^{-n} - e^a| \to 0$ as $n \to \infty$, which says that (e^{-n}) converges to e^a in the usual metric on $\mathbb{R}$. But we know that (e^{-n}) converges to 0 in the usual metric. Hence by uniqueness of limits $e^a = 0$. But there is no such real number a, so $(-n)$ cannot converge in $(\mathbb{R}, d_b)$. Hence $(\mathbb{R}, d_b)$ is not complete.

(c) For similar reasons this is not complete. For consider the sequence (n) in $\mathbb{R}$. The sequence $(\tan^{-1}(n))$ converges to $\pi/2$ in the usual sense as $n \to \infty$. So $(\tan^{-1}(n))$ is Cauchy in the usual sense, which says that (n) is d_c-Cauchy. But suppose that (n) converges to $a \in \mathbb{R}$ in the metric space $(\mathbb{R}, d_c)$. Then $(\tan^{-1}(n))$ converges to $\tan^{-1}(a)$ in the usual sense. By uniqueness of limits this gives $\tan^{-1}(a) = \pi/2$. But there is no such real number a. So (n) does not converge in $(\mathbb{R}, d_c)$, and this space is not complete.

17.6 We write d, d' for the metrics on X, X' respectively.

(a) Suppose that (x_n) is a Cauchy sequence in X, and let $\varepsilon > 0$. Since $f : X \to X'$ is uniformly continuous, there exists $\delta > 0$ such that $d'(f(x), f(x')) < \varepsilon$ whenever $d(x, x') < \delta$. Since (x_n) is Cauchy, there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \delta$ whenever $m > n \geqslant N$. So $d'(f(x_m), f(x_n)) < \varepsilon$ whenever $m > n \geqslant N$. This shows that $(f(x_n))$ is a Cauchy sequence in X'.

(b) Suppose that (y_n) converges to y in X'. Then by continuity of f^{-1} and Exercise 6.23, $(f^{-1}(y_n))$ converges to $f^{-1}(y)$ in X.

(c) Suppose that X' is complete and that (x_n) is a Cauchy sequence in X. Then $(f(x_n))$ is a Cauchy sequence in X' by (a). Since X' is complete, $(f(x_n))$ converges in X', so by (b) (x_n) converges in X. Thus X is complete.

(d) Now suppose that $f : X \to X'$ is a bijective map of metric spaces such that both f and f^{-1} are uniformly continuous. If X' is complete then so is X by (c) above. If X is complete then so is X' by (c) applied with f replaced by f^{-1}.

17.7 (a) Suppose that $D \subseteq \mathbb{R}$ and that $f : D \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leqslant K|x - y|^\alpha$ for some constants $K, \alpha > 0$ and all $x, y \in D$. Let $\varepsilon > 0$, and put $\delta = (\varepsilon/K)^{1/\alpha}$. Then whenever $x, y \in D$ and $|x - y| < \delta$ we have

$$|f(x) - f(y)| \leqslant K|x - y|^\alpha < K(\varepsilon/K)^{(1/\alpha)(\alpha)} = \varepsilon.$$

So f is uniformly continuous on D.

(b) This time we suppose that for any $x, y \in [a, b]$ we have $|f(x) - f(y)| \leqslant K|x - y|^\alpha$ for some constants K, α with $\alpha > 1$. Then for any $x \in [a, b]$ and $h \in \mathbb{R}$ with $h > 0$ and $x + h \in [a, b]$,

$$0 \leqslant \frac{|f(x+h) - f(x)|}{|h|} \leqslant K|h|^{\alpha-1}, \quad \text{and as } h \to 0 \text{ we have } |h|^{\alpha-1} \to 0 \text{ since } \alpha > 1.$$

This shows that f is differentiable on $[a, b]$ with derivative zero everywhere in $[a, b]$, so f is constant on $[a, b]$ (by the mean value theorem).

(c) By the mean value theorem, for any $x, y \in [a, b]$ we have $f(x) - f(y) = (x - y)f'(\xi)$ for some ξ between x and y. The conclusion follows immediately.

17.8 We note that $x^5 + 7x - 1 = 0$ iff $x = (1 - x^5)/7$. So define $f(x) = (1 - x^5)/7$. Then $f([0, 1]) \subseteq [0, 1]$: for $f'(x) = -5x^4/7 < 0$ for all $x \in [0, 1]$, and $f(0) = 1/7$, $f(1) = 0$. Also, $|f'(x)| \leqslant 5/7$ for all $x \in [0, 1]$, so by Exercise 17.7 (c), f is a contraction on $[0, 1]$ and by Theorem 17.22 it has a unique fixed point p in $[0, 1]$. This gives a solution of $x^5 + 7x - 1 = 0$ which is unique in $[0, 1]$. (Notice that the intermediate value theorem gives existence, but not uniqueness, of a solution in $[0, 1]$.)

17.9 The derivative of the cosine function is the negative sine function, and $0 \leqslant \sin x \leqslant \sin 1$ on $[0, 1]$. So $\cos x$ is decreasing on $[0, 1]$, with $\cos 0 = 1$ and $\cos 1 > 0$. So the cosine function does map $[0, 1]$ into $[0, 1]$. Moreover, since $|\sin x| \leqslant \sin 1 < 1$ for $x \in [0, 1]$, by Exercise 17.7 (c) the cosine function is a contraction on $[0, 1]$ and by Theorem 17.22 we can get successive approximations x_n to a solution of $\cos x = x$ by iterating the cosine function, beginning with $x_1 = 0.5$ say. This gives $x_2 = \cos x_1 = 0.88$, $x_3 = \cos x_2 = 0.64$, and successive x_n as $0.80, 0.70, 0.76, 0.72, 0.75, 0.73, 0.745, 0.735, 0.742, 0.737$. So it begins to look as if the unique solution of $\cos x - x$ in $[0, 1]$ is 0.74 correct to two decimal places. To check this, we calculate $\cos x - x$ at $x = 0.735$ and at 0.745, and get respectively positive and negative answers, so by the intermediate value theorem there must be a point between 0.725 and 0.745 where $\cos x - x = 0$. This checks that 0.74 is a solution of $\cos x = x$ correct to two decimal places. (There are faster algorithms for getting an approximate solution. For example the Newton-Raphson method with the same starting point is correct to two decimal places after only two stages.)

17.10 Let $f(x) = \sqrt{2 + \sqrt{x}}$. Then $f'(x) = \dfrac{1}{4\sqrt{x}\sqrt{2 + \sqrt{x}}}$ which is positive in $[\sqrt{3}, 2]$ so f is increasing there. Now $f(2) = \sqrt{2 + \sqrt{2}} < \sqrt{2 + 2} = 2$, and $f(2) = \sqrt{2 + \sqrt{2}} > \sqrt{2 + 1} = \sqrt{3}$. Also, $f(\sqrt{3}) = \sqrt{2 + 3^{1/4}}$ and $1 < 3^{1/4} < 2$, so $\sqrt{2 + 1} < \sqrt{2 + 3^{1/4}} < \sqrt{2 + 2} = 2$, so both $f(\sqrt{3})$ and $f(2)$ lie in $[\sqrt{3}, 2]$. This shows that f does map $[\sqrt{3}, 2]$ into itself. Also, for $x \in [\sqrt{3}, 2]$ we have $0 < f'(x) \leqslant \dfrac{1}{4.3^{1/4}\sqrt{2 + 3^{1/4}}} < 1$, so f is a contraction on $[\sqrt{3}, 2]$. Now by Banach's fixed point theorem, f has a unique fixed point p in $[\sqrt{3}, 2]$. Moreover the sequence defined in the question is obtained by iterating f, and we may think of the iteration as beginning with $x_2 = \sqrt{2 + \sqrt{2}}$, which lies in $[\sqrt{3}, 2]$. So by Banach's fixed point theorem the sequence converges to p. The point p is in $[\sqrt{3}, 2]$ and satisfies $p = \sqrt{2 + \sqrt{p}}$, so $p^2 = 2 + \sqrt{p}$, so $(p^2 - 2)^2 = p$, which says that p is a root of the equation $x^4 - 4x^2 - x + 4 = 0$.

17.11 Clearly f has no fixed point in $(0,\, 1/4)$ since the only solutions of $x^2 = x$ are $x = 0,\, 1$. Now f does map $(0,\, 1/4)$ into itself, since $0 < x^2 < 1/16$ when $0 < x < 1/4$. Moreover, for $x,\, y \in (0,\, 1/4)$ we have $|f(x) - f(y)| = |x^2 - y^2| = |(x - y)(x + y)| \leqslant |x - y|/2$, so f is a contraction of $(0,\, 1/4)$. But $(0,\, 1)$ is not complete, so this does not contradict Banach's fixed point theorem.

17.12 Since $[1,\, \infty)$ is a closed subspace of the complete space $\mathbb{R}$, it is complete by Proposition 17.7. Also, $f(x) = x + 1/x \geqslant 1$ for $x \geqslant 1$, so f maps $[1,\, \infty)$ into itself. Finally, $f'(x) = 1 - 1/x^2$, so for $x,\, y \in [1,\, \infty)$ the mean value theorem gives $|f(x) - f(y)| = |x - y||f'(\xi)|$ for some ξ between x and y, so $|f(x) - f(y)| < |x - y|$. [But we note that the above argument does not show that f is a contraction on $[1,\, \infty)$, since there is no $K < 1$ such that $1 - 1/\xi^2 \leqslant K$ for all $\xi \in [1,\, \infty)$.]

17.13 (a) By the contraction mapping theorem applied to $f^{(k)}$, there is a unique point $p \in X$ such that $f^{(k)}(p) = p$. Now consider $f(p)$. We have $f^{(k)}(f(p)) = f(f^{(k)})(p) = f(p)$, so $f(p)$ is a fixed point of $f^{(k)}$ and by uniqueness we must have $f(p) = p$. This says p is a fixed point of f itself. Moreover, if also $f(q) = q$ then by an easy induction we see that $f^{(n)}(q) = q$ for any $n \in \mathbb{N}$. In particular $f^{(k)}(q) = q$, so by uniqueness $q = p$. Hence f itself has a unique fixed point in X.

Now for some fixed $x \in X$ consider the sequence $(f^{(n)}(x))$. We wish to show that it converges to p. This is true for the subsequence $(f^{(nk)}(x))$ by the contraction mapping theorem, since $f^{(k)}$ is a contraction of X. Let K be a contraction constant for $f^{(k)}$. Let $\varepsilon > 0$. Since $(f^{(nk)}(x))$ converges to p, there exists $N_1 \in \mathbb{N}$ such that $d(f^{(nk)}(x),\, p) < \varepsilon/2$ for all $n \geqslant N_1$, where d is the metric on X. Now let $M = \max\{d(f(x),\, p),\, d(f^{(2)}(x),\, p),\, \ldots,\, d(f^{(k-1)}(x),\, p)\}$, and let $N_2 \in \mathbb{N}$ be such that $K^{N_2} M < \varepsilon/2$. (We may choose such an N_2 since $0 \leqslant K < 1$.) Put $N = \max\{N_1,\, N_2\}$. Then whenever $n \geqslant Nk$ we have $n = mk + i$ for some $m \geqslant N$ and some integer i satisfying $0 \leqslant i < k$. We get

$$d(f^{(n)}(x),\, p) \leqslant d(f^{(mk+i)}(x),\, f^{(mk)}(x)) + d(f^{(mk)}(x), p) \leqslant K^m d(f^{(i)}(x),\, x) + \varepsilon/2$$

$$\leqslant K^m M + \varepsilon/2 \leqslant K^{N_2} M + \varepsilon/2 < \varepsilon.$$

This shows that $(f^{(n)}(x))$ converges to p.

(b) The idea for seeing that $\cos : \mathbb{R} \to \mathbb{R}$ is not a contraction is that its derivative can get arbitrarily close to 1. Suppose for a contradiction that $\cos : \mathbb{R} \to \mathbb{R}$ is a contraction with contraction constant $K < 1$. Since $\sin x \to 1$ as $x \to \pi/2-$, we may choose $\delta > 0$ such that $\sin x > K$ for all $x \in [\pi/2 - \delta, \pi/2]$. By the mean value theorem $|\cos \pi/2 - \cos(\pi/2 - \delta)| = |\delta \sin \xi|$ for some $\xi \in (\pi/2 - \delta,\, \pi/2)$, so $|\cos \pi/2 - \cos(\pi/2 - \delta)| > K\delta$, contradicting the contraction condition.

To see that $\cos^{(2)} : \mathbb{R} \to \mathbb{R}$ is a contraction, put $f(x) = \cos(\cos x)$. Then $g'(x) = \sin x . \sin(\cos x)$, and for any $x \in \mathbb{R}$ we have

$$|g'(x)| \leqslant |\sin(\cos x)| \leqslant \sin 1 < 1,$$

so g is a contraction by Exercise 17.7 (c).

17.14 (a) Clearly d' satisfies (M1). By definition (M2) holds for d'. To check the triangle inequality $d'(a, c) \leqslant d'(a, b) + d'(b, c)$ for any three points $\{a, b, c\}$ we first note that this is trivial if any of the two points are equal, so it is enough to check it for three distinct points. Now $d'(x, z) = 2 < 3 = d'(x, y) + d'(y, z)$, $d'(x, y) = 2 < 3 = d'(x, z) + d'(z, y)$, and finally $d'(y, z) = 1 < 4 = d'(y, x) + d'(x, z)$. The other possibilities follow by symmetry.

Next we see that d' is Lipshitz equivalent to d. This follows for any two metrics on a finite set X, since for any distinct points $x, y \in X$ we may consider the ratio $d(x, y)/d'(x, y)$, and let k be the maximum of these ratios as x, y vary over all distinct pairs in X. Then $d(x, y) \leqslant kd'(x, y)$ for any pair $x, y \in X$ (trivially if $x = y$). Similarly there is a positive constant h such that $hd'(x, y) \leqslant d(x, y)$ for any pair $x, y \in X$.

(b) Since for example $1 = d(f(x), f(y)) = d(y, z) = 1$, and also $d(x, y) = 1$, f is not a d-contraction. However,

$d'(f(x), f(y)) = d'(y, z) = 1 = d'(x, y)/2,$
$d'(f(x), f(z)) = d'(y, z) = 1 = d'(x, z)/2,$
$d'(f(y), f(z)) = d'(z, z) = 0 \leqslant d'(y, z)/2,$

and the other contraction conditions follow by symmetry. So f is a d'-contraction with contraction constant $1/2$.

17.15 Consider the composition

$$X \overset{\Delta}{\to} X \times X \xrightarrow{1 \times f} X \times X \overset{d}{\to} \mathbb{R}.$$

Each map here is continuous (by Propositions 5.19, 5.22 and Exercise 5.17) so the composition F is continuous (by Proposition 5.18). Since X is compact, F attains its lower bound l on X. Suppose $l > 0$, and that l is attained at x_0, so $d(x_0, f(x_0)) = l$. Then x_0 and $f(x_0)$ are distinct, so by assumption $l = F(f(x_0)) = d(f(x_0), f(f(x_0))) < d(x_0, f(x_0)) = l$. This contradiction shows that l must be 0. Since l is attained, this says that $d(p, f(p)) = 0$ for some $p \in X$, so $f(p) = p$ and we have proved existence of a fixed point.

Uniqueness For distinct fixed points p, q we would have $d(p, q) = d(f(p), f(q)) < d(p, q)$, contradiction. So p is the unique fixed point.

17.16 (a) For $f_1,\ f_2 \in \mathcal{C}[0,\,1]$,

$$|I(f_1) - I(f_2)| = \left|\int_0^1 (f_1(x) - f_2(x))\,\mathrm{d}x\right| \leqslant \int_0^1 |f_1(x) - f_2(x)|\,\mathrm{d}x$$
$$\leqslant \sup_{x\in[0,\,1]} |f_1(x) - f_2(x)| = d_\infty(f_1,\ f_2),$$

so given $\varepsilon > 0$ take $\delta = \varepsilon$, and whenever $d_\infty(f_1,\ f_2) < \delta$ the above calculation shows that $|I(f_1) - I(f_2)| < \delta = \varepsilon$. This proves that I is continuous.

(b) We know that $\mathcal{C}[0,\,1]$ is complete (see Example 17.16) so we just need to show that F is a contraction of $\mathcal{C}[0,\,1]$. This has two ingredients: first, for any $y \in \mathcal{C}[0,\,1]$ we show that $F(y) \in \mathcal{C}[0,\,1]$. Then we show that F satisfies the contraction condition.

So let $y : [0,\,1] \to \mathbb{R}$ be continuous. We want to prove that $x \mapsto g(x) + \dfrac{1}{2}\displaystyle\int_0^1 \sin(xt)y(t)\,\mathrm{d}t$ is continuous. Since g is continuous, we just need to show that $x \mapsto G(x) = \dfrac{1}{2}\displaystyle\int_0^1 \sin(xt)y(t)\,\mathrm{d}t$ is continuous. Now by the mean value theorem, for any $t,\ x_1,\ x_2 \in [0,\,1]$ and for some ξ between x_1t and x_2t we have

$$|\sin(x_1t) - \sin(x_2t)| = |(x_1t - x_2t)\cos\xi| \leqslant |x_1 - x_2||t| \leqslant |x_1 - x_2|.$$

Hence

$$|G(x_1) - G(x_2)| = \left|\frac{1}{2}\int_0^1 (\sin(x_1t) - \sin(x_2t))y(t)\,\mathrm{d}t\right| \leqslant \left(\frac{1}{2}\int_0^1 |y(t)|\,\mathrm{d}t\right)|x_1 - x_2|.$$

Now given $\varepsilon > 0$ we may take $\delta = 2\varepsilon/\int_0^1 |y(t)|\,\mathrm{d}t$ and then $|G(x_1) - G(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$. This completes the proof that $F(y) \in \mathcal{C}[0,\,1]$ when $y \in \mathcal{C}[0,\,1]$.

Now for any $y_1,\ y_2 \in \mathcal{C}[0,\,1]$, $d_\infty(F(y_1),\ F(y_2)) = \sup_{x\in[0,\,1]} |F(y_1)(x) - F(y_2)(x)|$. But for all $x \in [0,\,1]$,

$$|F(y_1)(x) - F(y_2)(x)| = \frac{1}{2}\left|\int_0^1 \sin(xt)(y_1(t) - y_2(t))\ \mathrm{d}t\right| \leqslant \frac{1}{2}d_\infty(y_1,\ y_2).$$

Hence

$$d_\infty(F(y_1),\ F(y_2)) \leqslant \frac{1}{2}d_\infty(y_1,\ y_2),$$

and F is a contraction as required.