SOLUTIONS TO EXERCISES FOR

MATHEMATICS 145B — Part 2

Spring 2015

II. Constructing and deconstructing spaces

II.1 : Disjoint unions

Additional exercises

1. (i) Suppose first that X and Y are discrete. Then every subset of X is open and likewise for Y. Every subset of $X \amalg Y$ is given by a disjoint union $A \amalg B$ where $A \subset X$ and $B \subset Y$. Since X and Y are discrete, A and B are open subsets; by the definition of disjoint union topology this means that $A \amalg B$ is open in $X \amalg Y$. By the second sentence of this paragraph, this means that every subset of $X \amalg Y$ is open, and this in turn means that $X \amalg Y$ must be discrete.

Conversely, suppose that $X \amalg Y$ is discrete. Then for each pair of subsets $A \subset X$ and $B \subset Y$ we know that $A \amalg B$ is open. By the definition of disjoint union topology this means that A and B are open in X and Y respectively. Since these subsets are aribtrary, it follows that every subset of X is open and likewise for Y. This means that both X and Y are discrete spaces.■

(ii) Since a subspace of a Hausdorff space is Hausdorff and $X \amalg Y$ contains subspaces homeomorphic to X and Y, it follows that each of the latter is Hausdorff if $X \amalg Y$ is.

Conversely, suppose that X and Y are Hausdorff, and let p and q be distinct points in $X \amalg Y$. Then there are three cases to consider: The two points p and q could both lie in X, they could both lie in Y, or one could lie in X and the other in Y.

Suppose first that both point lie in X. Since X is Hausdorff there are disjoint open neighborhoods of U and V of p and q in X; the corresponding subsets $U \amalg \emptyset$ and $V \amalg \emptyset$ are then disjoint open neighborhoods of p and q in the disjoint union. Similarly, if both points lie in Y then disjoint neighborhoods in Y yield disjoint neighborhoods in $X \amalg Y$ (replace X with Y in the argument, and also replace $W \amalg \emptyset$ with $\emptyset \amalg W$).

Suppose now that one point lies in X and the other in Y. Without loss of generality we might as well assume that $p \in X$ and $q \in Y$ (switch the roles of p and q to get the other case). In this case the disjoint neighborhoods are merely $X \amalg \emptyset$ and $\emptyset \amalg Y$.■

(iv) A finite union of compact subsets is compact, so if X and Y are compact then so is $X \amalg Y$. Conversely, if $X \amalg Y$ is compact, then X and Y are compact because they are homeomorphic to closed subspaces of $X \amalg Y$.■

2. Follow the hint. The restrictions of the "identity" to $\left[0, \frac{1}{2}\right)$ and $\left[\frac{1}{2}, 1\right]$ are continuous because these are merely inclusion mappings. Therefore Proposition II.1.1 implies that the "identity" map is continuous. However, if the reverse "identity" map

$$[0,1] \quad \longrightarrow \quad \left[0, \tfrac{1}{2}\right) \amalg \left[\tfrac{1}{2}, 1\right]$$

were continuous, then the codomain would be compact and connected because it is the continuous image of the compact connected space $[0,1]$. Since the two pieces of the disjoint union are nonempty, the codomain is not connected; similarly, by (iv) in the preceding exercise it is not compact. Therefore the reverse "identity" map is not continuous and the mapping

$$\left[0, \tfrac{1}{2}\right) \amalg \left[\tfrac{1}{2}, 1\right] \quad \longrightarrow \quad [0,1]$$

is not a homeomorphism.■

3. (i) We shall first verify that $f \amalg g$ is continuous using Proposition II.1.1; in other words, we shall show that the restrictions to $X \amalg \emptyset \cong X$ and $\emptyset \amalg Y \cong Y$ are continuous. By definition the restriction to X is the composite of $f : X \to X'$ with the inclusion $X' \cong X \amalg \emptyset \subset X' \amalg Y'$, so it is a composite of continuous mappings and hence it is continuous. Likewise, the restriction to $\emptyset \amalg Y$ is the composite of $g : Y \to Y'$ with the inclusion $\emptyset \amalg Y' \cong Y' \subset X' \amalg Y'$, and hence this composite is also continuous. Therefore $f \amalg g$ is continuous by Proposition II.1.1.

Since f and g are homeomorphisms, they have continuous inverses, and by the reasoning of the preceding paragraph we know that $f^{-1} \amalg g^{-1}$ is continuous. We claim that

$$(f \amalg g)^{-1} \quad = \quad f^{-1} \amalg g^{-1} \ .$$

In other words, we need to check that the composites

$$(f \amalg g) \circ \left(f^{-1} \amalg g^{-1}\right) \ , \quad \left(f^{-1} \amalg g^{-1}\right) \circ (f \amalg g)$$

are the identity mappings on $X' \amalg Y'$ and $X \amalg Y$ respectively. Direct calculation shows that the first composite evaluated at $(x', 1)$ or $(y', 2)$ is equal to $(x', 1)$ or $(y', 2)$ respectively, and similarly for the second composite, and by the preceding sentence this implies that the continuous mappings are inverses of each other.■

(ii) Once again follow the hint. More precisely, we have

$$T_{Y,X} \circ T_{X,Y}(x,y) \quad = \quad T_{Y,X}(y,x) \quad = \quad (x,y) \ , \qquad T_{X,Y} \circ T_{Y,X}(y,x) \quad = \quad T_{X,Y}(x,y) \quad = \quad (y,x)$$

so that $T_{Y,X} \circ T_{X,Y}$ is the identity on $X \times Y$ and $T_{X,Y} \circ T_{Y,X}$ is the identity on $Y \times X$. Therefore the two mappings are inverse to each other.■

4. An open set in $X_1 \amalg X_2$ has the form $W_1 \amalg W_2$ where W_1 is open in X_1 and W_2 is open in X_2. Since $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for the respective topologies, we have $W_1 = \cup_\alpha U_\alpha$ where $U_\alpha \in \mathcal{B}_1 \cup \{\emptyset\}$, and we also have $W_2 = \cup_\beta V_\beta$ where $V_\beta \in \mathcal{B}_2 \cup \{\emptyset\}$. These mean that

$$W_1 \amalg W_2 \quad = \quad (\cup_\alpha U_\alpha \amalg \emptyset) \ \bigcup \ (\cup_\beta \emptyset \amalg V_\beta)$$

as asserted in the exercise.■

5. An explicit map from $(X \amalg Y) \times Z$ to $(X \times Z) \amalg (Y \times Z)$ is given by $\alpha(x, 1; z) = (x, z; 1)$ and $\alpha(y, 2; z) = (y, z; 2)$, and it is straightforward to verify that this map has an inverse sending $(x, z; 1)$ to $(x, 1; z)$ and $(y, z; 2)$ to $(y, 2; z)$. We need to show that α is continuous and open. It will suffice to show that images and inverse images of basic open subsets (under α) are open.

The basic open subsets of $(X \amalg Y) \times Z$ have the form $(U \amalg V) \times W$ where U, V, W are open in X, Y, Z respectively. The image of such an open set under α is given by $(U \times W) \amalg (V \times W)$, which is a basic open set in $(X \times Z) \amalg (Y \times Z)$. Therefore α sends open sets to open sets.

The preceding discussion also implies that the inverse image of the basic open set $(U \times W) \amalg (V \times W)$ under α is equal to the basic open set $(U \amalg V) \times W$, and hence the inverse image of a basic open subset under α is a basic open set.

Combining these, we see that the map α is a homeomorphism from $(X \amalg Y) \times Z$ to $(X \times Z) \amalg (Y \times Z)$.∎

II.2 : Quotient spaces

Problem from Munkres, § 22, pp. 144 – 145

4. (*a*) The hint describes a well-defined continuous map from the quotient space W to the real numbers. The equivalence classes are simply the curves $g(x, y) = C$ for various values of C, and they are parabolas that open to the left and whose axes of symmetry are the x-axis. It follows that there is a 1–1 onto continuous map from W to $\mathbb{R}$. How do we show it has a continuous inverse? The trick is to find a continuous map in the other direction. Specifically, this can be done by composing the inclusion of $\mathbb{R}$ in $\mathbb{R}^2$ as the x-axis with the quotient projection from $\mathbb{R}^2$ to W. This gives the set-theoretic inverse to $\mathbb{R}^2 \to W$ and by construction it is continuous. Therefore the quotient space is homeomorphic to $\mathbb{R}$ with the usual topology.∎

(*b*) Here we define $g(x, y) = x^2 + y^2$ and the equivalence classes are the circles $g(x, y) = C$ for $C > 0$ along with the origin. In this case we have a continuous 1–1 onto map from the quotient space V to the nonnegative real numbers, which we denote by $[0, \infty)$ as usual. To verify that this map is a homeomorphism, consider the map from $[0, \infty)$ to V given by composing the standard inclusion of the former as part of the x-axis with the quotient map $\mathbb{R}^2 \to V$. This is a set-theoretic inverse to the map from V to $[0, \infty)$ and by construction it is continuous.∎

Additional exercises

1. We claim that every subset of $X/\mathcal{R}$ is both open and closed. But a subset of the quotient is open and closed if and only if the inverse image has these properties, and every subset of a discrete space has these properties.∎

2. Let $\alpha : A \to A/\mathcal{R}_0$ and $\xi : X \to X/\mathcal{R}$ denote the quotient projections. Then $j : A/\mathcal{R}_0 \to X/\mathcal{R}$ is the unique mapping such that $j \circ \alpha = \xi | A$, and therefore it is continuous by Theorem II.2.1. The hypotheses imply that j is 1–1 and onto. Therefore it is only necessary to verify that j has a continuous inverse.

The hypotheses on q imply that if $u \, \mathcal{R} \, v$ then $q(u) \, \mathcal{R}_0 \, q(v)$, and by the definition of α this means that $\alpha \circ q(u) = \alpha \circ q(v)$. Therefore by Theorem II.2.1 there is a unique continuous mapping $r : X/\mathcal{R} \to A/\mathcal{R}_0$ such that $r \circ \xi = \alpha \circ q$. We then have

$$r \circ j \circ \alpha \quad = \quad r \circ \xi | A \quad = \quad \alpha \circ q | A \quad = \quad \alpha \circ \mathrm{id}_A \quad = \quad \alpha$$

which implies that $r \circ j$ and the identity on $A/\mathcal{R}_0$ agree on the image of α. Since α is onto, it follows that $r \circ j$ is the identity map for $A/\mathcal{R}_0$.

Since the mapping r is continuous, it will suffice to prove that r is an inverse function to j; by the preceding discussion, it only remains to prove that $j \circ r$ is the identity on $X/\mathcal{R}$. By the last sentence of the preceding paragraph, we know that $j \circ r \circ j = j = \mathrm{id}_{X/\mathcal{R}} \circ j$, which means that $j \circ r$

and the identity on $X/\mathcal{R}$ agree on the image of j. But we know that j is onto, and therefore it follows that $j \circ r$ is the identity map for $X/\mathcal{R}$. As noted previously, this completes the proof that $A/\mathcal{R}_0$ is homeomorophic to $X/\mathcal{R}$.■

3. (*a*) The relation is reflexive because $x = 1 \cdot x$, and it is reflexive because $y = \alpha x$ for some $\alpha \neq 0$ implies $x = \alpha^{-1} y$. The relation is transitive because $y = \alpha x$ for $\alpha \neq 0$ and $z = \beta y$ for $y \neq 0$ implies $z = \beta \alpha x$, and $\beta\alpha \neq 0$ because the product of nonzero real numbers is nonzero.■

(*b*) Use the hint to define q; we may apply the preceding exercise if we can show that for each $a \in S^2$ the set $q^{-1}(\{a\})$ is contained in an $\mathcal{R}$-equivalence class. By construction $q(v) = |v|^{-1}v$, so $q(x) = a$ if and only if x is a positive multiple of a (if $x = \rho\, a$ then $|x| = \rho$ and $q(x) = a$, while if $a = q(x)$ then by definition a and x are positive multiples of each other). Therefore if $x\mathcal{R}y$ then $q(x) = \pm\, q(y)$, so that $r(x)\mathcal{R}_0\, r(y)$ and the map

$$S^2/[x \equiv \pm x] \quad \longrightarrow \quad \mathbb{RP}^2$$

is a homeomorphism.■

4. Needless to say we shall follow the hints in a step by step manner.

Let $h : D^2 \to S^2$ be defined by

$$h(x, y) = (x, y, \sqrt{1 - x^2 - y^2}\,).$$

Verify that h preserves equivalence classes and therefore induces a continuous map $\overline{h}$ on quotient spaces.

To show that $\overline{h}$ is well-defined it is only necessary to show that its values on the $\mathcal{R}'$-equivalence classes with two elements are the same for both representatives. If $\pi : S^2 \to \mathbb{RP}^2$ is the quotient projection, this means that we need $\pi \circ h(u) = \pi \circ h(v)$ if $|u| = |v| = 1$ and $u = -v$. This is immediate from the definition of the equivalence relation on S^2 and the fact that $h(w) = w$ if $|w| = 1$.

Why is $\overline{h}$ a $1 - 1$ and onto mapping?

By construction h maps the equivalence classes of points on the unit circle onto the points of S^2 with $z = 0$ in a 1–1 onto fashion. On the other hand, if u and v are distinct points that are not on the unit circle, then $h(u)$ cannot be equal to $\pm\, h(v)$. The inequality $h(u) \neq -h(v)$ follows because the first point has a positive z-coordinate while the second has a negative z-coordinate. The other inequality $h(u) \neq h(v)$ follows because the projections of these points onto the first two coordinates are u and v respectively. This shows that $\overline{h}$ is 1–1. To see that it is onto, recall that we already know this if the third coordinate is zero. But every point on S^2 with nonzero third coordinate is equivalent to one with positive third coordinate, and if $(x, y, z) \in S^2$ with $z > 0$ then simple algebra shows that the point is equal to $h(x, y)$.

Finally, prove that $\mathbb{RP}^2$ is Hausdorff and $\overline{h}$ is a closed mapping.

If the first statement is true, then the second one follows because the domain of $\overline{h}$ is a quotient space of a compact space and continuous maps from compact spaces to Hausdorff spaces are always closed. Since $\overline{h}$ is already known to be continuous, 1–1 and onto, this will prove that it is a homeomorphism.

So how do we prove that $\mathbb{RP}^2$ is Hausdorff? Let v and w be points of S^2 whose images in $\mathbb{RP}^2$ are distinct, and let P_v and P_w be their orthogonal complements in $\mathbb{R}^3$ (hence each is a 2-dimensional vector subspace and a closed subset). Since Euclidean spaces are Hausdorff, we can find an $\varepsilon > 0$ such that $N_\varepsilon(v) \cap P_v = \emptyset$, $N_\varepsilon(w) \cap P_w = \emptyset$, $N_\varepsilon(v) \cap N_\varepsilon(w) = \emptyset$, and $N_\varepsilon(-v) \cap N_\varepsilon(w) = \emptyset$. If T denotes multiplication by -1 on $\mathbb{R}^3$, then these conditions imply that the four open sets

$$N_\varepsilon(v), \quad N_\varepsilon(w), \quad N_\varepsilon(-v) = T\left(N_\varepsilon(v)\right), \quad N_\varepsilon(-w) = T\left(N_\varepsilon(w)\right)$$

are pairwise disjoint. This implies that the images of the distinct points $\pi(v)$ and $\pi(w)$ in $\mathbb{RP}^2$ lie in the disjoint subsets $\pi\left[N_\varepsilon(v)\right]$ and $\pi\left[N_\varepsilon(w)\right]$ respectively. These are open subsets in $\mathbb{RP}^2$ because their inverse images are given by the open sets $N_\varepsilon(v) \cup N_\varepsilon(-v)$ and $N_\varepsilon(w) \cup N_\varepsilon(-w)$ respectively.■

5. One physical model for the quotient space construction is to pinch the top edge of the solid square to the center point $\left(\frac{1}{2}, 1\right)$ (it might be helpful to draw a picture of this process). We can do this continuously by linearly shrinking each horizontal segment $[0,1] \times \{y\}$ in the square to the segment centered at $\left(\frac{1}{2}, y\right)$ whose length is equal to $1 - y$; in other words, at level y the linear function sends 0 to $\frac{1}{2}\,y$ and 1 to $1 - \frac{1}{2}\,y$. Here is the explicit formula:

$$h(x,y) \quad = \quad \left(\tfrac{1}{2}\,y + x(1-y), y\right) .$$

By construction, this function is continuous.

One can now check directly that h maps the solid square onto the solid triangle, it sends the top edge of the square to the top vertex of the triangle, and it is 1–1 on the set of all points satisfying $y < 1$.

The preceding discussion shows that h induces a continuous 1–1 onto map $\overline{h}$ from $X/\mathcal{R}$ to the closed triangular region. Since X is compact, the quotient is also compact, and since the triangular region is a subset of $\mathbb{R}^2$, it is Hausdorff. Since a 1–1 onto continuous map from a compact space to a Hausdorff space is a homeomorphism, it follows that $\overline{h}$ is a homeomorphism from the quotient space to the solid triangular region.■

6. Let $\pi : X \to X/\mathcal{E}$ be the quotient projection, and let U and V be open neighborhoods of the equivalence classes $[(0,1)]$ and $[(0,2)]$ in $X/\mathcal{E}$. We shall prove that $U \cap V$ is nonempty.

Since π is continuous, there are positive numbers $\delta_1, \delta_2 > 0$ such that π maps $(-\delta_1, \delta_1) \times \{1\}$ into U and also maps $(-\delta_2, \delta_2) \times \{2\}$ into V. Let δ be the smaller of δ_1 and δ_2. Then the images of $(-\delta, \delta) \times \{1\}$ and $(-\delta, \delta) \times \{2\}$ under the quotient map are the same; in particular, we have

$$\pi\left(\tfrac{1}{2}\,\delta, 1\right) \quad = \quad \pi\left(\tfrac{1}{2}\,\delta, 2\right) .$$

Since the left hand side belongs to $\pi^{-1}[U]$ and the right hand side belongs to $\pi^{-1}[V]$, it follows that the given point lies in $U \cap V$ and therefore the latter must be nonempty. In fact, the intersection contains infinitely many points, for $\frac{1}{2}\delta$ can be replaced by any positive real number $\eta < \delta$.■

Note. It might be useful to compare this with the more intuitive description of the example on page 52 of Crossley (see Example 4.37).

7. Let $p : X \times Y \to X$ be projection onto the first coordinate. Then $u\mathcal{R}v$ implies $p(u) = p(v)$ and therefore there is a unique continuous map $X \times Y/\mathcal{R} \to X$ sending the equivalence class of (x,y) to x. Set-theoretic considerations imply this map is 1–1 and onto, and it is a homeomorphism because p is an open mapping.■