SOLUTIONS TO EXERCISES FOR

MATHEMATICS 145B — Part 3

Spring 2015

III. Homotopy

III.1 : Basic definitions

Problem from Munkres, § 51*, pp.* 330

1. It will suffice to consider two special cases:

(A) The special case where $h = h'$.

(B) The special case where $k = k'$.

If we can verify the conclusion in these two cases, the general conclusion follows, for (A) implies that $k \circ h \simeq k' \circ h$ and (B) implies that $k' \circ h \simeq k' \circ h'$. By the transitivity of homotopy, these yield the desired relation $k' \circ h' \simeq k \circ h$.

SUBCASE (A). Let $M : Y \times [0,1] \to Z$ be a homotopy from k to k', and set $N(x,t) = M(h(x),t)$. Then N is a homotopy from $k \circ h$ to $k' \circ h$.

SUBCASE (B). Let $P : X \times [0,1] \to Y$ be a homotopy from h to h', and set $Q(x,t) = k' \circ P(x,t)$. Then P is a homotopy from $k' \circ h$ to $k' \circ h'$.■

2. (a) In fact, every map is homotopic to the constant map whose value everywhere is zero, and an explicit homotopy from an arbitrary map $f : X \to [0,1]$ is given by $H(x,t) = (1-t) \cdot f(x)$.■

(b) We shall first show that if f and g are constant maps then they are homotopic. Suppose that $f(x) = p$ and $g(x) = q$ for all $x \in X$. Since X is arcwise connected, there is a continuous curve $\gamma : [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$. A homotopy between f and g is then given by a homotopy of constant mappings $H(x,t) = \gamma(t)$.

Next, we show that every continuous mapping $[0,1] \to X$ is homotopic to a constant map. Given $\varphi : [0,1] \to X$, the homotopy $K(x,t) = \varphi(x - tx)$ is a homotopy from φ to the constant map with value $\varphi(0)$.

Now the second paragraph implies that every map is homotopic to a constant map, and the first implies that all constant maps are homotopic. By transitivity, all maps must be homotopic.■

3. (a) More generally, every nonempty convex subset of $\mathbb{R}^n$ is homotopic to a constant by a straight line homotopy

$$H(x,t) \quad = \quad (1-t)\,x \; + \; t\,p_0$$

where p_0 is an arbitrary point in the convex set.■

(b) Let $H : X \times [0,1] \to X$ be a homotopy from the identity to a constant map with value p. Then for each $x \in X$ the curve $\gamma(t) = H(x,t)$ joins x to p, and therefore every point in X lies in the same arc component as p, which means that X must be arcwise connected.■

(c) The contractibility hypothesis implies that the identity map on Y is homotopic to a constant map, say with value p. Call this map C_p. Then we have $\mathrm{id}_Y \simeq C_p$, and by Exercise 1 (see above) this yields

$$f \quad = \quad \mathrm{id}_Y \circ f \quad \simeq \quad C_p \circ f$$

where the right hand side is merely the constant map $X \to Y$ whose value is p. Hence every continuous mapping f is homotopic to this constant map, so that there is only one homotopy class in $[X, Y]$.■

(d) The contractibility hypothesis implies that the identity map on X is homotopic to a constant map, say with value x. Call this map K_x, so that $\mathrm{id}_X \simeq K_x$. Applying Exercise 1 once again, we obtain the relationships

$$f \quad = \quad f \circ \mathrm{id}_X \quad \simeq \quad f \circ K_x$$

where the right hand side is merely the constant map $X \to Y$ whose value is $f(x)$. Therefore every continuous mapping from X to Y is homotopic to some constant map.

Since Y is arcwise connected, we can now use the reasoning in Exercise 2(b) to conclude that two constant maps are always homotopic. By transitivity it follows that all maps from X to Y are homotopic if X is contractible and Y is arcwise connected.■

NOTE. If Y is not arcwise connected, then $[X, Y]$ is in 1–1 correspondence with the set of arc components for Y (see below).

Additional exercises

1. Continuous maps from P to X are completely determined by their values at the unique point of P, so the continuous maps from P to X are in 1–1 correspondence with the points of X. Also, a homotopy between two such maps corresponds to a continuous curve starting at one point and ending at the other, so two maps from P to X are homotopic if and only if their values at the unique point can be joined by a curve in X. In other words, the functions are homotopic if and only if their values lie in the same arc component of X.■

2. A continuous map sends a connected component of X to a connected component of Y. Since the components of Y are single points, it follows that a continuous map from the connected space X to the discrete space Y must be constant. Therefore the set of continuous maps from X to Y is in 1–1 correspondence with the points of Y. If two of these constant maps are homotopic, then their values in Y can be joined by a continuous curve, so that these points lie in the same arc component. Since Y is discrete, each arc component consists of exactly one point, so if two continuous mappings from X to Y are homotopic they must be equal; in other words, we have a 1–1 correspondence $[X, Y] \cong Y$.■

3. (i) The definition of star convexity implies that the straight line homotopy

$$H(x, t) \quad = \quad (1 - t)\, x \; + \; t\, a_0$$

lies in A, so if $k : A \to \{a_0\}$ is the constant map and $i : \{a_0\} \to A$ is inclusion, then $i \circ k$ is homotopic to the identity.■

(ii) Let $x \in \cup_\alpha \ A_\alpha$; then we need to prove that all points on the closed line segment defined by

$$(1-t)\,x \ + \ t\,p \qquad\qquad (0 \ \leq \ \ t \ \leq \ 1)$$

lie in $\cup_\alpha \ A_\alpha$.

If x belongs to the union, then there is some indexing variable γ such that $x \in A_\gamma$. Since A_γ is convex, we know that $(1-t)\,x \ + \ t\,p \in A_\gamma$ for all $t \in [0,1]$. The conclusion of the exercise now follows because $A_\gamma \subset \cup_\alpha \ A_\alpha$.■

(iii) Each of the coordinate axes is convex. If u and v lie on the $x-$axis, then their second coordinates are zero and hence the same is true for $(1-t)u + tv$, where $t \in \mathbb{R}$; a similar argument works for points on the $y-$axis, which consists of all points whose first coordinates are zero. We can now apply (ii) and conclude that the union Y of the axes is a star convex set, for the origin lies in both axes.

To see that the union of the axes is not convex, it suffices to consider the points

$$(1-t)\,\mathbf{e}_1 \ + \ t\,\mathbf{e}_2$$

for $0 < t < 1$, where the $\mathbf{e}_i$ are the standard unit vectors in $\mathbb{R}^2$. Both coordinates are nonzero, so this point does not lie in Y, but both of the unit vectors lie in Y.■

4. We need to show that if $f \simeq f'$ and $g \simeq g'$, then $g' \circ f' \simeq g \circ f$. Since this statement is the conclusion for Exercise 1in Munkres (see above), it follows that the homotopy class $v \circ u$ does not depend upon the choices for f and g.■

5. (i) Once again, all we have to show is that the homotopy class of $f \circ g$ does not change if we replace g by a homotopic mapping g'; in other words, we need to know that $g \simeq g'$ implies $f \circ g \simeq f \circ g'$. But this follows from the preceding exercise and/or Exercise 1 from Munkres.

Suppose now that f is the identity map on Y. Then $f \circ g = g$, so on the level of homotopy classes we have $f_*(v) = v$, which is the value of the identity map on $[Y, Y]$ takes at v.■

(ii) This just follows from the associative rule for composition. Let $v = [\alpha]$ be a homotopy class on which f_* is defined. Then we have

$$(h \circ f)_*(v) \quad = \quad [(h \circ f) \circ \alpha] \quad = \quad [h \circ (f \circ \alpha)] \quad = \quad h_*(f_*(v)) \quad = \quad h_* \circ f_*(v)$$

which verifies the identity stated in the exercise.■

(iii) This time we need to know that if $g \simeq g'$ then $g \circ f \simeq g' \circ f$, and once again Exercise 1 from Munkres yields this property. Furthermore, if f is the identity then $f^*([g]) = [g \circ f] = [g]$, which is the value that the identity map on $[Y, Y]$ takes at $[g]$.■

(iv) This time we have

$$(f \circ h)^*(v) \quad = \quad [\alpha \circ (f \circ h)] \quad = \quad [[\alpha \circ f) \circ h] \quad = \quad h^*(f^*(v)) \quad = \quad h^* \circ f^*(v)$$

which verifies the identity stated in the exercise.■

III.2 : Homotopy eqivalence

Problem from Munkres, § 53, pp. 366 – 367

1. Since A is a strong deformation retract of B and B is a strong deformation retract of X there exist homotopies

$$H : B \times [0,1] \to B \qquad K : X \times [0,1] \to X$$

Such that $H|B \times \{0\}$ and $K|X \times \{0\}$ are given by identity maps, the images of $H|B \times \{1\}$ and $K|X \times \{1\}$ lie in A and B respectively, and the restrictions $H|A \times [0,1]$ and $K|B \times [0,1]$ send (a,t) and (b,t) to a and b respectively (*i.e.*, the homotopies are fixed on A and B respectively). Let $\rho : X \to B$ be given by $K|X \times \{1\}$, and concatenate the homotopies by setting $L(x,t)$ equal to $K(x,2t)$ if $t \le \frac{1}{2}$ and equal to $K(\rho(x), 2t-1)$ if $t \ge \frac{1}{2}$. One can check directly that the two formulas yield the same result if $t = \frac{1}{2}$, so one obtains a well defined homotopy from these formulas.

The assumptions on H and K imply that $L|X \times \{0\}$ is given by the identity and $L|X \times \{1\}$ lies in A. Furthermore, since $\rho(a) = a$ if $a \in A$, it follows that $L|A \times [0,1]$ sends (a,t) to a. Therefore L is a strong deformation retraction from X to A.■

3. A space X is homotopic to itself, with $f : X \to X$ and its homotopy inverse given by the identity mapping. If $f : X \to Y$ is a homotopy equivalence and $g : Y \to X$ is its homotopy inverse, then the identities $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$ imply that g is a homotopy equivalence and f is its homotopy inverse. Finally, if $f : X \to Y$ and $f' : Y \to Z$ are homotopy equivalences, let $g' : Z \to Y$ and $g : Y \to X$ be homotopy inverses. Then we have the homotopy composition identities

$$[g \circ g'] \circ [f' \circ f] \;=\; [(g \circ g') \circ (g \circ f)] \;=\; [g \circ (g' \circ (f' \circ f))] \;=$$

$$[g \circ ((g' \circ f') \circ f)] \;=\; [g \circ (\mathrm{id}_Y \circ f)] \;=\; [g \circ f] \;=\; [\mathrm{id}_X]$$

and if we switch the roles of X and Y, f and g, and f' and g' in this derivation we obtain the companion identity

$$[f' \circ f] \circ [g \circ g'] \;=\; [\mathrm{id}_Z]$$

and therefore $f' \circ f : X \to Z$ is a homotopy equivalence.■

Additional exercises

1. Follow the hint and consider the composites $g^{-1} \circ f \circ f^{-1}$ and $g^{-1} \circ g \circ f^{-1}$. Since $f \simeq g$, these two maps are homotopic. Therefore we have

$$g^{-1} \;=\; g^{-1} \circ f \circ f^{-1} \;\simeq\; g^{-1} \circ g \circ f^{-1} \;=\; f^{-1}$$

and hence the inverses to f and g are homotopic.■

2. If two discrete spaces have the same cardinalities, then every 1–1 correspondence from one to the other is a homeomorphism because every map from a discrete space is continuous. Since homeomorphic spaces are homotopy equivalent, it follows that two discrete spaces with the same cardinalities are homotopy equivalent.

Conversely, suppose that two discrete spaces X and Y are homotopy equivalent, and let $f : X \to Y$ be a homotopy equivalence. If I denotes the unit interval, then the map $f_* : [I, X] \to [I, Y]$ is an isomorphism, for if g is a homotopy inverse then $g_* \circ f_*$ and $f_* \circ g_*$ are identity maps, so that $g_* = (f_*)^{-1}$. Now if D is a discrete space, then there is a 1–1 correspondence from $[I, D]$ to D because the image of a continuous mapping $I \to D$ is contained in a connected component of D and all components of D are one point subset. Therefore if f is a homotopy equivalence then there is a 1–1 correspondence between $X \cong [I, X]$ and $Y \cong [I, Y]$.∎

3. Follow the hint. Let $i : X \to X \times Y$ be a slice inclusion sending x to (x, y_0), and let $p : X \times Y \to X$ be the coordinate projection. Then we have

$$\mathrm{id}_X \;=\; p \circ i \;=\; p \circ \mathrm{id}_{X \times Y} \circ i \;\simeq\; \text{constant} \circ i$$

because $X \times Y$ is contractible. The right hand side is a constant map, and therefore it follows that id_X is homotopic to a constant map. But this means that X is contractible. If we switch the roles of X and Y, then a similar argument shows that Y is also contractible.∎

4. Follow the hint, and let $\rho : \mathbb{R}^{n+1} - \{\mathbf{0}\} \to S^n$ be the map sending v to the unit vector $|v|^{-1} \cdot v$. The hypothesis on f and g implies that $\mathbf{0}$ does not lie in the image of the straight line segment joining $f(x)$ to $g(x)$, where $x \in S^n$ is arbitrary (if the two points coincide, the curve is just the constant map). Therefore $j \circ f$ and $j \circ g$ are homotopic maps from S^n to $\rho : \mathbb{R}^{n+1} - \{\mathbf{0}\}$. Now $\rho \circ j$ is the identity on S^n, and therefore it follows that $f = \rho \circ j \circ f \simeq \rho \circ j \circ g = g$, which is the statement in the conclusion of the exercise.∎

5. Follow the hint, and show that the image of the homotopy $H(z, t) = z + ta$ does not contain 0; since H_0 is the standard circle centered at 0 and H_1 is the circle centered at a, the conclusion of the exercise will follow from this.

So suppose that $0 = H(z, t) = z + ta$. Then we have $z = ta$ where $|z| = 1$, $t \in [0, 1]$ and $|a| < 1$. Therefore we have $1 = |z| = |ta| = |t| \cdot |a| \leq 1 \cdot 1 = 1$, a contradiction. The source of the problem is the assumption that $H(z, t) = 0$ has a solution, and hence no solution can exist.∎

6. Let g_1 and g_2 be homotopy inverses to f_1 and f_2 respectively; we claim that

$$(g_1 \times g_2) \circ (f_1 \times f_2) \;\simeq\; \mathrm{id}_{X_1 \times X_2} , \qquad (f_1 \times f_2) \circ (g_1 \times g_2) \;\simeq\; \mathrm{id}_{Y_1 \times Y_2}$$

which implies that the original product map is a homotopy equivalence.

Let p_1, p_2 denote the projections of $X_1 \times X_2$ onto X_1 and X_2 respectively, and let q_1, q_2 denote the projections of $Y_1 \times Y_2$ onto Y_1 and Y_2 respectively.

> DIGRESSION. Suppose we are given homotopic continuous mappings $h_1, k_1 : Z \to W_1$ and $h_2, k_2 : Z \to W_2$. Let $\lambda : Z \to W_1 \times W_2$ and $\mu : Z \to W_1 \times W_2$ be the continuous mappings whose coordinate projections are given by (h_1, h_2) and (k_1, k_2) respectively. CLAIM: *We have $\lambda \simeq \mu$.*
>
> PROOF OF CLAIM. Let $N_1 : Z \times [0, 1] \to W_1$ and $N_2 : Z \times [0, 1] \to W_2$ be the homotopies, and let $\mathbf{N} : W \to Z_1 \times Z_2$ be the map whose coordinate projections are N_1 and N_2 respectively. Then $\mathbf{N}$ is a homotopy from λ to μ.

We shall use the digression as one step in following the hint. The definitions of the product maps $f_1 \times f_2$ and $g_1 \times g_2$ immediately yield the following identities (verify each of them!):

$$q_1 \circ (f_1 \times f_2) \;=\; f_1 \circ p_1 , \quad q_2 \circ (f_1 \times f_2) \;=\; f_2 \circ p_2$$

$$p_1 \circ (g_1 \times g_2) \quad = \quad g_1 \circ q_1 \;, \quad p_2 \circ (g_1 \times g_2) \quad = \quad g_2 \circ q_2$$

These identities and the homotopy inverse conditions imply several additional identities:

$$p_i \circ (g_1 \times g_2) \circ (f_1 \times f_2) \quad = \quad g_i \circ f_i \circ p_i \quad \simeq \quad p_i \quad = \quad p_i \circ \mathrm{id}_{X_1 \times X_2} \qquad (i = 1, 2)$$

$$q_i \circ (f_1 \times f_2) \circ (g_1 \times g_2) \quad = \quad f_i \circ g_i \circ q_i \quad \simeq \quad q_i \quad = \quad q_i \circ \mathrm{id}_{Y_1 \times Y_2} \qquad (i = 1, 2)$$

We can now apply the discussion in the digression to conclude that

$$(g_1 \times g_2) \circ (f_1 \times f_2) \quad \simeq \quad \mathrm{id}_{X_1 \times X_2} \;, \qquad (f_1 \times f_2) \circ (g_1 \times g_2) \quad \simeq \quad \mathrm{id}_{Y_1 \times Y_2}$$

which is what we needed to prove.■

7. (i) Let $H : F \times [0,1] \to F$ be strong deformation retraction data, so that $H|F \times \{0\}$ is given by the identity, the image of $F \times \{1\}$ is contained in B and $H(b,t) = b$ for all $b \in B$ and $t \in [0,1]$. Extend H to $H^* : X \times [0,1] \to X$ by setting $H^*(x,t) = x$ if $x \in A$; this extends H because $B = A \cap F$. By construction, $H^*|X \times \{0\}$ is given by the identity, the image of $X \times \{1\}$ is contained in A and $H^*(a,t) = b$ for all $a \in A$ and $t \in [0,1]$.■

(ii) For $i = 0$ and 1 let $H^{(i)} : F_i \times [0,1] \to F_i$ be strong deformation retraction data, so that $H^{(i)}|F_i \times \{0\}$ is given by the identity, the image of $F_i \times \{1\}$ is contained in C and $H(c,t) = c$ for all $c \in C$ and $t \in [0,1]$. The last condition implies that we can define deformation retraction data $K : X \times [0,1] \to X$ such that $K|F_i \times [0,1] = H^{(i)}$.■

8. We shall begin by verifying the first assertion. If X is contractible, then id_X is homotopic to a constant map C. Therefore if $f : X \to Y$ is continuous we have $f = f \circ \mathrm{id}_X \simeq f \circ C$, which is a constant map. Conversely, if every continuous map $f : X \to Y$ is homotopic to a constant mapping, then this is true for $f = \mathrm{id}_X$, and the preceding statement implies that X is contractible.

We now prove the second assertion. As before, if X is contractible, then id_X is homotopic to a constant map C. Therefore if $f : Y \to X$ is continuous we have $f = \mathrm{id}_X \circ f \simeq C \circ f$, which is a constant map. Conversely, if every continuous map $f : Y \to X$ is homotopic to a constant mapping, then this is true for $f = \mathrm{id}_X$, and the preceding statement implies that X is contractible.■

9. (i) The arc components of a space W are given by $[P, W]$, where $P = \{p\}$. If f is a homotopy equivalence, then the argument in a previous exercise implies that $f_* : [P, X] \to [P, Y]$ is an isomorphism.■

(ii) Let g be a homotopy inverse to f. Then each arc component $A_\alpha \subset X$ maps into an arc component $B_\alpha \subset Y$ under f, and similarly each arc component $B_\beta \subset Y$ maps into an arc component $A_\beta \subset X$ under g. Combining these, we see that each arc component $A_\alpha \subset X$ maps to some possibly different arc component $A_{\gamma(\alpha)} \subset X$.

We claim that this component is just A_α. Since $g \circ f$ is homotopic to the identity, it follows that $g \circ f$ and the identity map A_α to the same arc component $A_{\gamma(\alpha)}$. But clearly the identity maps A_α to itself, and therefore the same must be true for $g \circ f$.

By the preceding we have continuous mappings of arc components $f_\alpha : A_\alpha \to B_\alpha$ and $g_\alpha : B_\alpha \to A_\alpha$. It will suffice to show that the homotopies $H : g \times f \simeq \mathrm{id}_X$ and $K : f \times g \simeq \mathrm{id}_Y$ map A_α to B_α and B_α to A_α. Once again, we know that the images of H and K are contained in arc components of X and Y respectively. Since one end of each homotopy is the identity, the arc components of the images must be the same as the original arc components. Therefore the homotopies H and K pass to homotopies $H_\alpha : A_\alpha \times [0,1] \to A_\alpha$ and $K_\alpha : B_\alpha \times [0,1] \to B_\alpha$, which means that f_α is a homotopy equivalence.■

(iii) This is essentially the argument of (i) with "connected component(s)" replacing "arc component(s)." The main issue is to develop some formalism for dealing with sets of connected components of topological spaces and their behavior under continuous mappings.

If $g : A \to B$ is a continuous map, then for each connected component C of A we know that the connected set $g[A]$ must lie in a connected component of B, and therefore if $\mathbf{ConnComp}(E)$ denotes the set of connected components of a space E, then the continuous mapping g induces a map of sets

$$g_{**} \;=\; \mathbf{ConnComp}(g) : \mathbf{ConnComp}(A) \;\longrightarrow\; \mathbf{ConnComp}(B)$$

and it is an elementary exercise to verify that this construction has the following properties:

If g is the identity map on X, then g_{**} is the identity map on $\mathbf{ConnComp}(X)$.

If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $(g \circ f)_{**} = g_{**} \circ f_{**}$.

If $f, g : X \to Y$ are homotopic, then $f_{**} = g_{**}$.

The verifications of the first two are similar to the corresponding arguments for arc components, and likewise for the third statement but with a few simple modifications. Suppose that C is a connected component of X, and let h be a homotopy from f to g. Since $C \times I$ is connected (where I denotes the closed unit interval), it follows that h maps $C \times I$ into a component of Y. By the choice of h, its restrictions to $X \times \{0\}$ and $X \times \{1\}$ send $C \times \{0\}$ and $C \times \{1\}$ into $f_{**}(C)$ and $g_{**}(C)$ respectively; therefore both are contained in the connected component which contains the image of h, which means that $f_{**}(C)$ and $g_{**}(C)$ must be equal to this connected component. Since C was an arbitrary connected component, it follows that $f_{**} = g_{**}$ as claimed.

Once we have this, we can argue as in the case of arc components to conclude that if f is a homotopy equivalence and h is a homotopy inverse, then f_{**} and h_{**} must be isomorphisms.■

(iv) Since every arc component is contained in a connected component, for every space S we have a natural projection

$$\alpha_S : [P.S] \;\longrightarrow\; \mathbf{ConnComp}(S)$$

which sends an arc component of a space S to the connected component which contains it. It is straightforward to check that this construction is natural with respect to continuous maps; namely, if $f : X \to Y$ is continuous, then $\alpha_Y \circ f_* = f_{**} \circ \alpha_X$. Graphically such a relationship is often formulated as a statement that the following diagram of functions is commutative (all composites with the same source and target are equal):

$$\begin{tikzcd}
{[P,X]} \arrow[r, "f_*"] \arrow[d, "\alpha_X"] & {[P,Y]} \arrow[d, "\alpha_Y"] \\
\mathbf{ConnComp}(X) \arrow[r, "f_{**}"] & \mathbf{ConnComp}(Y)
\end{tikzcd}$$

If f is a homotopy equivalence, then the results in the first two paragraphs imply that f_* and f_{**} are isomorphisms. Therefore, if α_X is an isomorphism then α_Y is also an isomorphism with inverse $f_{**} \circ \alpha_X \circ (f_*)^{-1}$, and if α_Y is an isomorphism then α_X is also an isomorphism with inverse $f_* \circ \alpha_X^{-1} \circ (f_{**})^{-1}$ (note that the diagram is helpful for finding the correct fomulas for the inverse maps).■

III.3 : The circle

Additional exercises

1. (a) This space is homotopy equivalent to the circle, for if $\simeq$ denotes homotopy equivalence we have $S^1 \simeq S_1$ and $\{0\} \simeq D^2$.$\blacksquare$

(b) This space is also homotopy equivalent to the circle, for we have $S^1 \simeq S_1$ and $\{0\} \simeq [0,1]$.$\blacksquare$

(c) This space is also homotopy equivalent to the circle, for we have $S^1 \simeq S_1$ and $\{0\} \simeq \mathbb{R}$.$\blacksquare$

(d) This space is contractible, for it is equal to $[1,\infty) \times \mathbb{R}$ and both spaces are homotopy equivalent to $\{0\}$.$\blacksquare$

(e) This space is also contractible, for it is equal to $(1,\infty) \times \mathbb{R}$ and both spaces are homotopy equivalent to $\{0\}$.$\blacksquare$

(e) This space is also contractible, for it is equal to $(-\infty,1) \times \mathbb{R}$ and both spaces are homotopy equivalent to $\{0\}$.$\blacksquare$

(f) This space is also homotopy equivalent to the circle. Let E be the circle, and let F be the open ray; both are closed subsets of the space X under consideration. The intersection of the two sets is the one point set P consisting of $(1,0)$ and P is a strong deformation retract of $\mathbb{R}^+ \times \{0\}$. Therefore E is a strong deformation retract of X by a previous exercise.$\blacksquare$

(g) This space is also homotopy equivalent to the circle. In fact, if $E \subset X$ is the inclusion of the circle as before, we can show that E is a strong deformation retract of X. Consider the deformation retraction data $H : \left(\mathbb{R}^2 - \{\mathbf{0}\}\right) \to S^1$ given by the homotopy which pushes everything to the circle along lines through the origin. This homotopy sends $X \times [0,1]$ into X, and the associated map $X \times [0,1] \to X$ defines the required deformation retraction data.$\blacksquare$

(h) NOT ASSIGNED, BUT A SLIGHTLY MORE CHALLENGING QUESTION OF THE SAME TYPE. *Answer the same question for* $S^1 \cup [0,\infty) \times \mathbb{R}$.

Once again the space has the homotopy type of a circle, but the definition of the deformation retraction data is more complicated and depends upon whether $v \in \mathbb{R}^2$ satisfies $|v| \leq 1$ or $|v| \leq 1$; if $|v| = 1$ then both definitions will simply send v to itself.

If $|v| \geq 1$ we use the same deformation retraction data as before:

$$H(v,t) \quad = \quad (1-t)\,v \;+\; t \cdot |v|^{-1} \cdot v$$

Suppose now that $|v| \leq 1$. If $|v| = 1$ then we must have $H(v,t) = v$, so the only issue is to define this map when $v = (x,y)$ with $y \geq 0$ and to check that this definition yields (x,y) if $|v| = 1$. Geometrically, we want to squash points in the half disk $\{x \geq 0\ ,\quad |v| \leq 1\}$ horizontally into the semicircle $\{x \geq 0\ ,\quad |v| = 1\}$ such that points on the latter are left untouched. The algebraic/analytic formula for this homotopy is given as follows:

$$H(x,y;t) \quad = \quad (1-t)\cdot(x,y) \;+\; t \cdot \left(\sqrt{1-y^2},\, y\right)$$

One can now check that these formulas yield deformation retraction data for the inclusion $E \subset X$.$\blacksquare$

2. (*a*) Let $\alpha, \beta : [0,1] \to \mathbb{R}$ be liftings of the closed curves $f \circ \theta, g \circ \theta$, where $\theta = p|[0,1]$. If $\oplus$ denotes the algebraic sum of α and β as real valued functions, then by the hint we know that $p \circ (\alpha \oplus \beta) = (f \circ \theta) \cdot (g \circ \theta)$. Therefore by the definition of degree in the exercises we have

$$\deg(f \cdot g) \quad = \quad \alpha \oplus \beta(1) \ - \ \alpha \oplus \beta(1) \quad =$$

$$(\alpha(1) + \beta(1)) \ - \ (\alpha(0) + \beta(0)) \quad = \quad (\alpha(1) - \alpha(0)\beta(1)) \ + \ (\beta(1) + \beta(0))$$

which simplifies to $\deg(f) + \deg(g)$.■

(*b*) Let $h : [0,1] \times [0,1] \to S^1$ be a homotopy from $f \circ \theta$ to $g \circ \theta$, and let H be a covering homotopy for h. Since the restrictions of h to $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$ are equal it follows that for each t the difference $H(1,t) - H(0,t)$ is an integer. Since this difference is a continuous integer valued function of t, it follows that it must be constant for all t. In particular, this means that

$$\deg(f) \quad = \quad H(1,0) \ - \ H(0,0) \quad = \quad H(1,1) \ - \ H(0,1) \quad = \quad \deg(g)$$

which is what we wanted to prove.■

(*c*) Since the degree only depends upon the homotopy class, for each m and n it will suffice to verify the identity for a pair of maps f_m, f_n where the degree of f_d is equal to d. The simplest representatives are the power maps $f_d(z) = z^d$, and the simplest choices of liftings are the maps $L_d(t) = dt$. Since $f_m \circ f_n = f_{mn}$, it follows that the lifing for the composite is merely L_{mn}. Therefore the degree of $f_{mn} = f_m \circ f_n$ is equal to $L_{mn}(1) = mn$, which is just $deg(f_m) \cdot \deg(f_n)$.■

3. The problem lies with the expression z^{t+1}. There is no way of defining such a function continuously over the entire nonzero complex plane unless $t \geq -1$ is an integer. It is possible to define such maps over large open subsets U by formulas of the form $\exp(t\,L(z))$ where L is a "branch of $\log z$" which can be defined over U, but it is not possible to define this on all of $\mathbb{C} - \{0\}$. In fact, if this were possible, the construction **would** define a homotopy between the two maps in the exercise.■

III.4 : The Brouwer Fixed Point Theorem

Additional exercises

1. (*a*) Suppose that X is not connected, and let A and B be nonempty open and closed subsets such that $X = A \cup B$ and $A \cap B = \emptyset$. Pick $a_0 \in A$ and $b_0 \in B$, and define $f : X \to X$ by $f(x) = a_0$ if $x \in B$ and $f(x) = b_0$ if $x \in A$. Then f is continuous, but f has no fixed points because $f[A] \cap A = f[B] \cap B = \emptyset$.■

(*b*) Let $h : X \to X$ be a continuous map which does not have a fixed point. Then $h \times \mathrm{id}_Y : X \times Y \to X \times Y$ also has no fixed points because $h \times \mathrm{id}_Y(x,y) = (x,y)$ would imply $h(x) = x$.■

2. Follow the hint. Since $X \times Y$ has the fixed point property, there is some (x,y) such that

$$(x,y) \quad = \quad h \times \mathrm{id}_Y(x,y) \quad (h(x),\, y)$$

and if we equate coordinates we conclude that $h(x) = x$. Similarly, if $g : Y \to Y$ is continuous, then $\mathrm{id}_X \times g$ has a fixed point, and this yields a fixed point for g.■

3. We need to use the hint and Exercise III.2.4 somehow. Suppose that f is not homotopic to the identity, and let $h(z) = -z$. Geometrically, this map is a $180°$ rotation, and it is homotopic to the identity through rotations by the formula $h(z,t) = \exp(\pi\, i\, t) \cdot z$. It then follows that $h \circ f \simeq f$ is not homotopic to the identity. Therefore the contrapositive of Exercise III.2.4 that there is some point z_0 such that $-z_0 = h \circ f(z_0) = -f(z_0)$, or equivalently $z_0 = f(z_0)$.■

NOTE. This is a special case of a result known as the **Lefschetz Fixed Point Theorem**. As noted in Munkres, there are analogous concepts of degrees for maps of higher dimensional spheres, and the Lefschetz Fixed Point Theorem generalizes to maps of such spheres as follows: If $f : S^n \to S^n$ is continuous and $\deg(f) \neq (-1)^{n+1}$, then f has a fixed point.