Summary of the key steps

(1) Reformulate problem into solving $g(x)=x$.

(2) Find a reasonable interval $[a,b]$ such that g maps $[a,b]$ into itself, and

(3) g has a continuous derivative such that $|g'(x)| \le c < 1$ on $[a,b]$.

THEN $y = \lim_{n\to} g^n(x)$ satisfies $g(y)=y$.

Sutherland, Thm. 17.26
Prop. 17.9
+ Mean Value Thm.

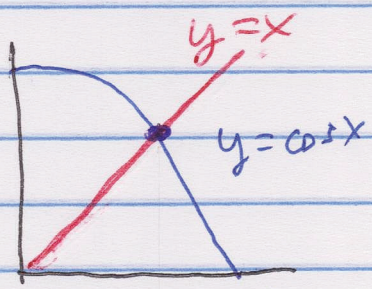
Another example

↓

Solve $\cos x = x$, $0 \le x \le \frac{\pi}{2}$.

$x=0 \Rightarrow \cos x = 1 > x$.

$x=\frac{\pi}{3} \Rightarrow \cos x = \frac{1}{2} < x$



graphic solution

Also $\frac{\pi}{3} > 1$. Hence cos maps $[0, \frac{\pi}{3}]$ to itself, and $\frac{d}{dx}\cos x = -\sin x \Rightarrow$

$$\left|\frac{d}{dx}\cos x\right| \le \frac{\sqrt{3}}{2} < 1 \text{ on } \left[0, \frac{\pi}{3}\right]$$

Hence we can solve $\cos x = x$ by taking the limit of the sequence $x_0 = 1$, $x_{n+1} = \cos x_n$. Here are some successive approximations:

x_5 0.701368724...
x_{10} 0.744237355...
x_{15} 0.738369204...
x_{20} 0.739184400...
x_{25} 0.739071365...
x_{30} 0.739087043...
x_{35} 0.739084868...
x_{40} 0.739085170...
x_{45} 0.739085128...
x_{50} 0.739085134...
x_{55} 0.739085133...

and the approximations after this are the same to the 9th decimal place.