Supplement to Chapter 13 of Sutherland,

Introduction to Metric and Topological Spaces (Second Edition)

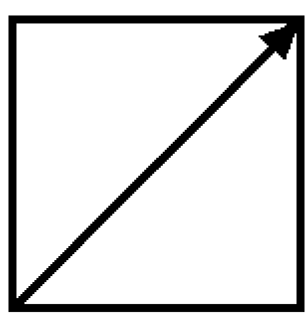
Continuous images of closed intervals

If $f: [a, b] \to \mathbb{R}$ is a continuous function, then the results in Chapters 12 and 13 of Sutherland imply that the image of $[a, b]$ under f is some closed interval, and more generally if $f: X \to Y$ is a continuous map defined on a compact connected space X, then $f[X]$ is compact and connected. However, even if we take X to be a closed interval in the real line, then there is a very broad range of possibilities for the image, including many examples which are not homeomorphic to each other and a some which are unexpected. For this reason it is important to be careful and not assume too much about a continuous image of a closed interval.

Square – filling curves

To illustrate the preceding discussion, we shall (informally) describe a continuous curve $\gamma: [0, 1] \to \mathbb{R}^2$ whose image is the unit square $[0, 1] \times [0, 1]$. At first this may seem impossible, because we normally think of curves as having no width, and intuitively it seems clear that something with zero width should not cover up an entire square. The basic problem with this reasoning is that the curves one normally works with are built out of pieces which have some sorts of differentiability properties, possibly over small subintervals (think about the boundary of a regular polygon); in fact, one can prove that the set of points traced out by such curves will have zero area, so that the image is an "infinitesimally small" portion of the square, and thus a square – filling curve cannot have any reasonable differentiability properties at most if not all points. Examples of such curves were first discovered by G. Peano (1858 – 1932), and subsequently many other examples of a similar nature were constructed.

Construction of the example. The square – filling curve γ is a limit of broken line curves. We may start with a straight line segment joining $(0, 0)$ to $(1, 1)$, which has a slope of $+1$.



The next broken line in the sequence of approximations is the zigzag curve on the left hand side of the following illustration, which is modified from page 43 of the book by Sagan in the list of references (at the end of this note).

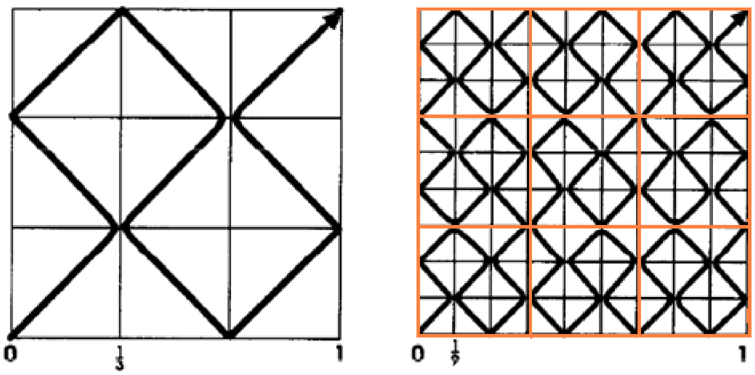


Fig. 3.6.2. First and Second Approximating Polygon for the Peano Curve with the Corners Rounded Off

The illustration also indicates how we obtain the next approximation. In the first approximation we have a finite sequence of segments whose slopes are $+1$ or -1. To form the next approximation, we replace each of these segments with a shrunken copy of the first broken line or a congruent copy of this, depending upon the positions of the first and last points on the curve. For example, we take a copy of the original broken line in the lower left square, we take a shrunken copy of its mirror image in with respect to the $y-$axis on the middle left square, and in the middle square we take a shrunken copy of the curve rotated by 180 degrees. Continuing in this fashion, we obtain a new approximation at the right, which is formed by stringing together $81 = 9^2$ line segments whose slopes are $+1$ or -1.

We can now perform the same construction on each of these pieces, obtaining a new approximation, which is formed by stringing together $6561 = 81^2 = 9^4$ line segments whose slopes are $+1$ or -1, and in fact we can continue this procedure to obtain a sequence of approximations such that at step k the curve is formed by stringing together 81^k line segments whose slopes are $+1$ or -1.

Next, we can then use uniform convergence theorems from real analysis to conclude that these approximations converge to a limit. Furthermore, for each of the approximations from step k onwards the image of the limit curve will contain the centers of all the 81^k squares which arise in the construction at step k (this is apparent from comparing each of the squares in the left hand drawing with their counterparts on the right). If we let C_k denote the set of all these centers, then the limit curve will contain the union of the sets C_k. Since the union of these subsets is dense in the unit square and the image of the curve is a closed bounded set, it follows that ***the image of the curve is the entire unit square*.**

We can elaborate upon this example to obtain continuous maps $[0, 1] \to \mathbb{R}^n$, where n is an arbitrary positive integer, such that the image is equal to the hypercube $[0, 1]^n$. In particular, if $n = 3$ we can do this using the composite

$$[0, 1] \to [0, 1]^2 \to [0, 1]^2 \times [0, 1]$$

where the first map is the square $-$ filling curve γ and the second sends (s, t) to $(\gamma(s), t)$.

The preceding discussion was deliberately informal in nature. A more rigorous account of the construction can be found in the references listed below:

H. Sagan, *Space – Filling Curves*. Springer – Verlag, New York, **1994**. (See Chapter **3**.)

J. R. Munkres, *Topology* (Second Edition). Prentice – Hall, Saddle River NJ, **2000**. (See Section **44**.)

The following film from the **1970**s contains animations for similar approximations which yield square – filling curves (the picture quality is primitive by today's standards but the illustrations are good, and the presentation is bland but well – organized).

http://www.youtube.com/watch?v=2e8QJBkCwvo

Continuous images of closed intervals

In view of the preceding examples, it would be enlightening to have simply stated criteria for recognizing whether a given topological space can be the continuous image of a closed interval. The definitive result on this question is due to H. Hahn (1879 – 1934) and S. Mazurkiewicz (1888 – 1945), and it dates back to the second decade of the 20th century.

HAHN – MAZURKIEWICZ THEOREM. *If X is a topological space, then X is the image of a continuous function defined on the closed unit interval $[0, 1]$ if and only if X is a compact, connected and locally connected metrizable space.*

A space X is said to be ***locally connected*** if for each point p in X and each open set U containing p there is an open connected set V such that p is in V and V is contained in U. Open sets in $\mathbb{R}^n$ are fundamental examples of locally connected sets (in fact, given a point p and an open set U containing it, one can find a ***convex*** open subset V containing p). It turns out that a space can be connected without being locally connected and vice versa (to see the latter, take the union of two open intervals in the real line whose associated closed intervals are disjoint).

Here are two textbook references for the preceding theorem:

J. G. Hocking and G. S. Young, *Topology*. Dover Publications, Mineola NY, **1988**. (See Section **3 – 5**.)

S. Willard, *General Topology*. Dover Publications, Mineola NY, **2004**. (See Section **31**.)

In connection with the preceding theorem, we should also mention the following very surprising result which is used in the course of the proof.

THEOREM. *Every compact metric space is the continuous image of a continuous map defined on the Cantor set.*

The two corresponding references for the proof of this result are Section $3 - 5$ in Hocking and Young and Section 30 in Willard.