Further results

Prop 13.15. X compact top. space, $f: X \to Y$ continuous $\Rightarrow f[X]$ compact

(The continuous image of a compact set is compact.)

Proof. Suppose $\mathcal{U} = \{U_\alpha\}$ is an open covering of $f[X]$. Write $U_\alpha = V_\alpha \cap f[X]$ where V_α is open in Y. Then the sets

$$f^{-1}[V_\alpha] = f^{-1}[V_\alpha \cap f[X]] = f^{-1}[U_\alpha]$$

form an open covering of $X \Rightarrow$ there is a finite subcovering $f^{-1}[V_{\alpha_1}], \ldots, f^{-1}[V_{\alpha_m}]$, or equivalently $f^{-1}[U_{\alpha_1}], \ldots, f^{-1}[U_{\alpha_m}]$.

Since $U_\alpha \subseteq f[X]$ we have $f[f^{-1}[U_\alpha]] = U_\alpha$, and therefor $U_{\alpha_1}, \ldots, U_{\alpha_m}$ are a finite subcovering of $f[X]$. ∎

Cor. $K \subseteq \mathbb{R}^n$ closed + bded, $f: K \to \mathbb{R}$ continuous $\Rightarrow f$ takes a maximum and minimum value. ∎

An inverse function theorem.

Thm 13.26 X compact, Y Hausdorff, $f: X \longrightarrow Y$ continuous 1-1 onto $\Longrightarrow$ f is a homeomorphism.

Proof. Enough to show that E closed in $X \Rightarrow f[E]$ closed in Y.

(This implies that f is open, for V open in X $\Longrightarrow X - V$ closed in $X \Rightarrow Y - f[V] = f[Y-V]$ is closed in $Y \Rightarrow f[V]$ open in Y.)

since f is 1-1 onto

But E closed in $X \Longrightarrow E$ compact $\Longrightarrow$ $f[E]$ compact in Y. Since Y is Hausdorff, this means that $f[E]$ is closed in Y. ∎

See inverse-fcn-thms.pdf for other situations in which f is 1-1 onto continuous $\Longrightarrow f$ is a homeomorphism.

Compactness and closed subsets

Thm. A top space X is compact $\Leftrightarrow$:

For each family $\mathfrak{O}$ of closed sets $\{F_\alpha\}$

Finite Intersection Property [such that each finite intersection

$F_{\alpha_1} \cap \ldots \cap F_{\alpha_m} \neq \emptyset$, we have $\bigcap_{\alpha \in \mathfrak{O}} F_\alpha \neq \emptyset$.

Cor. Suppose X is compact and we have a sequence $\{F_n\}$ of NONEMPTY closed subsets such that $F_1 \supseteq F_2 \supseteq \ldots$, then $\bigcap_n F_n \neq \emptyset$.

Proof of the Corollary. Show $\{F_n\}$ has the finite intersection property, then apply the theorem. But $F_{\alpha_1} \cap \ldots \cap F_{\alpha_m} =$

$F_{MAX\{\alpha_1, \ldots, \alpha_m\}}$. ∎

Proof of the theorem.

① Suppose X is compact, and let $\{F_\alpha\}$ have the finite intersection property. If $\bigcap_\alpha F_\alpha = \emptyset$, then $\mathfrak{U} = \{X - F_\alpha\}$ is an open covering.

If $X-F_{\alpha_1}, \ldots, X-F_{\alpha_n}$ is a finite subcovering, then $X = \cup X - F_{\alpha_j} = X - \bigcap_j F_{\alpha_j} \Rightarrow F_{\alpha_1} \cap \cdots \cap F_{\alpha_n} = \phi$. $\boxed{\text{CONTRADICTION}}$

The source of the contradiction was the assumption that $\bigcap_\alpha F_\alpha = \phi$, and hence $\bigcap_\alpha F_\alpha$ must be nonempty. ∎

② Conversely, suppose the condition in the theorem is satisfied. Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of X, let $F_\alpha = X - U_\alpha$. Then

$$\cap F_\alpha = \cap X - U_\alpha = X - \underbrace{\bigcup_\alpha U_\alpha}_{=X} = \phi.$$

The condition in the theorem says this cannot happen if every $F_{\alpha_1} \cap \cdots \cap F_{\alpha_m} \neq \phi$, so some finite intersection of this type is nonempty. But if $F_{\alpha_1} \cap \cdots \cap F_{\alpha_n} = \phi$, then as above we have $X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_m}$, so that $\mathcal{U}$ has a finite subcovering. Hence X is compact. ∎