# Addenda to 0.6

These are a few items which were mentioned in the lectures but not in `notes0.6.pdf`.

**COROLLARY 0.6A.** *If a topological space $X$ is arcwise connected, then it is connected.*

**Proof.** Fix $x_0 \in X$. Then for each $y \in X$ there is a continuous curve $\gamma_y : [0,1] \to X$ joining $x_0$ to $y$. Since $[0,1]$ is connected, it follows that the set $A_y = \gamma_y[[0,1]]$ is also connected. This means that $y$ lies in the same connected component as $x_0$. But $y$ was arbitrary, so this means that every point lies in the connected component containing $x_0$.■

Note that an example at the end of `notes0.6.pdf` shows that the reverse implication is false; there are connected spaces which are not arcwise connected. The example also shows that the closure of an arcwise connected subspace need not be arcwise connected.

In the lectures there were two examples of spaces which had connected components which were not open. One was the rationals, and the other was the set of all points on the real line which are either 0 or have the form $1/n$ for some positive integer $n$. One assertion in each discussion was that the connected components in each example were one point sets. This is acturally not too difficult to verify, so we shall present the argument here. Everything is based upon the following simple result:

**PROPOSITION 0.6.8.** *Let $A$ be a connected subset of the topological space, and let $C$ be an open and closed subset of $X$. Then either $A \subset C$ or $A \subset X - C$.*

**Proof.** The intersections $A \cap C$ and $A - C = A \cap (X - C)$ are open and closed subsets of $A$, so by the connectedness of $A$ we know that either $A \cap C = \emptyset$, in which case $A \subset X - C$, or else $A \cap C = A$, in which case $A \subset C$.■

**COROLLARY 0.6.9.** *Let $X$ be a topological space with the following property:* For each pair of points $x, y \in X$ there is an open and closed subset $C_{x,y}$ which contains $x$ but not $y$. *Then every connected component of $X$ contains exactly one point.*

Note that this property holds for both of the examples in the lectures.

> Consider the set of all points on the real line which are either 0 or have the form $1/n$ for some positive integer $n$. Each of the one point sets $\{1/n\}$ is open and closed in this subspace (with respect to the subspace topology), and since 0 is the only other point, it follows that the condition of the corollary is satisfied. Therefore the connected components are simply the one point subsets; all of these sets except $\{0\}$ are both open and closed, but $\{0\}$ is not open in this subspace.
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> Consider the rational numbers, and suppose that $x \neq y \in \mathbb{Q}$. Choose $h > 0$ such that
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> $$h = \frac{|x-y|}{\sqrt{2}} .$$
>
> Then $(x - h, x + h)$ is both open and closed in $\mathbb{Q}$ (it is closed because the points $x \pm h$ do not lie in $\mathbb{Q}$), and it does not contain $y$.

**Proof of Corollary 0.6.9.** Give $x \in X$, let $A$ be the connected component of $x$ in $X$, and if $y \neq x$ let $C_{x,y}$ be an open and closed subset which contains $x$ but not $y$. By Proposition 0.6.8 we must have $A \subset C_{x,y}$. This means that if $y \neq x$ then $y \notin A$, and the latter forces the conclusion $A = \{x\}$.■