Solutions to Chapter 14 exercises

14.1 Consider the sequence $(1/n)$ in $(0, 1)$. This has no subsequence converging to a point of $(0, 1)$ since the sequence $(1/n)$, and hence every subsequence, converges in $\mathbb{R}$ to 0.

14.2 Suppose for a contradiction that the sequentially compact metric space (X, d) is not bounded. Choose any point $x_0 \in X$. Then for any $n \in \mathbb{N}$ there exists a point in X, call it x_n, with $d(x_n, x_0) \geqslant n$. The sequnce (x_n) has no convergent subsequence, since any subsequence (x_{n_r}) is unbounded ($d(x_{n_r}, x_0) \geqslant n_r$). Hence X must be bounded.

14.3 Let A be a closed subset of a sequentially compact metric space X. Let (x_n) be any sequence in A. Then (x_n) is also a sequence in X, which is sequentially compact, so there is a convergent subsequence (x_{n_r}). The point this converges to must lie in A since A is closed in X (see Corollary 6.30). Hence A is also sequentially compact.

14.4 Let A be a sequentially compact subspace of a metric space X, and let $x \in \overline{A}$. Then (see Exercise 6.26) there is a sequence (a_n) of points in A converging to x. Since A is sequentially compact, there is some subsequence (a_{n_r}) of (a_n) converging to a point in A. But every subsequence of (a_n) converges to x, so $x \in A$. This tells us that A is closed in X (see Proposition 6.11 (c)).

14.5 Let (y_n) be a sequence in $f(X)$. For each $n \in \mathbb{N}$ there exists a point $x_n \in X$ such that $y_n = f(x_n)$. Since X is sequentially compact, there is some subsequence (x_{n_r}) of (x_n) which converges to a point $x \in X$. Then by continuity of f the subsequence $(y_{n_r}) = (f(x_{n_r})$ converges in Y to $f(x)$ (see Exercise 6.25). Hence $f(X)$ is sequentially compact.

14.6 This follows from Exercise 14.5. For if $f : X_1 \to X_2$ is a homeomorphism and X_1 is sequentially compact then so is X_2 by Exercise 14.5, since f is continuous and onto. Since the inverse of f is also continuous and onto, it follows likewise that if X_2 is sequentially compact then so is X_1.

14.7 This follows from Exercises 14.5 and 14.2. For if $f : X \to Y$ is a continuous map of metric spaces and X is sequentially compact, then by Exercise 14.5 so is $f(X)$, and hence, by Exercise 14.2, $f(X)$ is bounded.

14.8 By Exercise 14.7 the function f is bounded, so its bounds do exist. Now $f(X)$ is a sequentially compact subspace of $\mathbb{R}$ by Exercise 14.5. Hence $f(X)$ is closed in $\mathbb{R}$ by Exercise 14.4. But the bounds of a non-empty closed subset of $\mathbb{R}$ are in the set by Exercise 6.9. This says that the bounds of $f(X)$ are in $f(X)$, which means that they are attained.

14.9 Suppose that (X, d_X), (Y, d_Y) are sequentially compact metric spaces. In $X \times Y$ we shall use the product metric d_1: recall that $d_1((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$. Let $((x_n, y_n))$ be any sequence in $X \times Y$. First, since X is sequentially compact there is a subsequence (x_{n_r}) of (x_n) converging to a point $x \in X$. Now consider the sequence (y_{n_r}) in Y. Since Y is sequentially compact, there exists a subsequence $(y_{n_{r_s}})$ of (y_{n_r}) converging to a point $y \in Y$. Then $(x_{n_{r_s}})$ is a subsequence of (x_{n_r}) hence also converges to x. Consider the subsequence $((x_{n_{r_s}}, y_{n_{r_s}}))$ of $((x_n, y_n))$. This converges to (x, y): for let $\varepsilon > 0$. Since (x_{nr_s}) converges to x, there exists $S_1 \in \mathbb{N}$ such that $d_X(x_{nr_s}, x) < \varepsilon/2$ whenever $s \geqslant S_1$. Similarly there exists $S_2 \in \mathbb{N}$ such that $d_Y(y_{nr_s}, y) < \varepsilon/2$ whenever $s \geqslant S_2$. Put $S = \max\{S_1, S_2\}$. If $s \geqslant S$ then

$$d_1((x_{nr_s}, y_{n_{r_s}}), (x, y)) = d_X(x_{nr_s}, x) + d_Y(y_{nr_s}, y) < \varepsilon.$$

So $((x_n, y_n))$ has a subsequence converging to a point in $X \times Y$. This shows that $X \times Y$ is sequentially compact. (As we have seen, any 'product metric' will give the same answer.)

14.10 Suppose that the result is true for some $n \geqslant 1$, and let X be a bounded closed subset of $\mathbb{R}^{n+1}$. Then $X \subseteq [a, b]^{n+1}$ for some $a, b \in \mathbb{R}$, (by Exercise 5.7), and it is sufficient to prove that $[a, b]^{n+1}$ is sequentially compact, since X is closed in this space hence then also sequentially compact by Exercise 14.3. Now $[a, b]^n$ and $[a, b]$ are sequentially compact by inductive assumption and the allowed case $n = 1$ respectively, so $[a, b]^{n+1} = [a, b]^n \times [a, b]$ is sequentially compact by Exercise 14.9.

14.11 Let $x_n \in V_n$ for each $n \in \mathbb{N}$. Since X is sequentially compact, there is a subsequence (x_{n_r}) of (x_n) converging to some point $x \in X$. Since the V_n are nested, $x_{n_r} \in V_m$ for all r such that $n_r \geqslant m$. But V_m is closed in X, so $x \in V_m$ (by Corollary 6.30). This is true for all $m \in \mathbb{N}$, so $x \in \bigcap_{n=1}^{\infty} V_n$ and this intersection is non-empty.

14.12 Suppose that C is relatively compact in a metric space (X, d), and recall that for present purposes this means that $\overline{C}$ is sequentially compact. Now any sequence in C is also a sequence in $\overline{C}$, so it has a convergent subsequence. (In fact this subsequence converges to some point in $\overline{C}$).

Conversely suppose that every sequence in C has a convergent subsequence. We wish to show that $\overline{C}$ is sequentially compact. Let (x_n) be any sequence in $\overline{C}$. For each $n \in \mathbb{N}$, since $x_n \in \overline{C}$ there exists $y_n \in C$ such that $d(y_n, x_n) < 1/n$. Now consider the sequence (y_n) in C. By hypothesis this has a convergent subsequence (y_{n_r}), say converging to y. By Proposition 6.29, $y \in \overline{C}$. Now given any $\varepsilon > 0$ there exists $R_1 \in \mathbb{N}$ such that $d(y_{n_r}, y) < \varepsilon/2$ whenever $r \geqslant R_1$ and there exists $R_2 \in \mathbb{N}$ such that $1/n_r < \varepsilon/2$ whenever $r \geqslant R_2$. Put $R = \max\{R_1, R_2\}$. If $r \geqslant R$ then

$$d(x_{n_r}, y) \leqslant d(x_{n_r}, y_{n_r}) + d(y_{n_r}, y) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus any sequence in $\overline{C}$ has a subsequence converging to a point in $\overline{C}$ - in other words $\overline{C}$ is sequentially compact, so C is relatively compact.

14.13 The exercise does most of this! Following as suggested, we shall prove inductively that $[a, a_i] \subseteq A$ for $i = 1, 2, \ldots, a_n = b$. This is true for $i = 1$ since $a_0 = a \in A$, and since $a_1 - a_0 < \varepsilon$ where ε is a Lebesgue number for the cover $\{A, B\}$, we know that $[a_0, a_1]$ is contained in a single set of the cover, and this must be A since $A \cap B = \emptyset$. Suppose inductively that $[a, a_i] \subseteq A$ for some $i \in \{1, 2, \ldots, n-1\}$. Then we can repeat the above argument with a replaced by a_{n-1} and deduce that also $[a_{n-1}, a_n] \subseteq A$. Hence $[a, b] \subseteq A$, so $\{A, B\}$ is not a partition of $[a, b]$ after all. So $[a, b]$ is connected.

14.14 If $U_i = X$ for some $i \in \{1, 2, \ldots, n\}$ then any $\varepsilon > 0$ is a Lebesgue number for $\mathcal{U}$, since for any $\varepsilon > 0$, any set of diameter at most ε is contained in X and hence in U_i.

(i) Suppose now that $C_i \neq \emptyset$ for every $i \in \{1, 2, \ldots, n\}$. Then continuity of the function $f_i : X \to \mathbb{R}$ defined by $f_i(x) = d(x, C_i)$ follows from Exercise 6.16 (c). Also, from the definition it follows that all the values of $f_i(x)$ are non-negative.

(ii) Continuity of f follows from continuity of each f_i and Proposition 5.17. Let $x \in X$. Since $\mathcal{U}$ is a cover for X, $x \in U_i$ for at least one $i \in \{1, 2, \ldots, n\}$ so x is not in $C_i = X \setminus U_i$. Now C_i is closed in X, so $f_i(x) = d(x, C_i) > 0$ (by Exercise 6.16 (a)). But also $f_j(x) \geqslant 0$ for all $j \in \{1, 2, \ldots, n\}$ so $f(x) > 0$ as required.

(iii) By sequential compactness of X and Exercise 14.8, there exists $\varepsilon > 0$ such that $f(x) \geqslant \varepsilon$ for all $x \in X$.

(iv) Since there are just n values $d(x, C_i)$ it is clear that

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i) \leqslant \max\{d(x, C_i) : i \in \{1, 2, \ldots, n\}\}.$$

(v) For a given $x \in X$ let $\max\{d(x, C_i) : i \in \{1, 2, \ldots, n\}\} = d(x, C_{k(x)})$. We prove that $B_\varepsilon(x) \subseteq U_{k(x)}$ where ε is as in (iii) above. For suppose $d(y, x) < \varepsilon$. Then $\varepsilon \leqslant f(x) \leqslant d(x, C_{k(x)})$ so $d(y, x) < d(x, C_{k(x)})$. This says $d(y, x)$ is less than the distance from x to $C_{k(x)} = X \setminus U_{k(x)}$, so $y \in U_{k(x)}$. Hence $B_\varepsilon(x) \subseteq U_{k(x)}$ as required. It follows that for any $x \in X$ there is a set $U \in \mathcal{U}$ such that $B_\varepsilon(x) \subseteq U$, so ε is a Lebesgue number for the cover $\mathcal{U}$.

14.15 If say V_{n_0} is empty, then $\bigcap_{n=1}^{\infty} V_n = \emptyset$, whose diameter is 0 by definition. Likewise in this case diam $V_{n_0} = 0$ so $\inf\{\text{diam } V_n : n \in \mathbb{N}\} = 0$ also.

Suppose now that all the V_n are non-empty. (We already know from Exercise 14.11 that their intersection is non-empty.) Now $\bigcap_{n=1}^{\infty} V_n \subseteq V_m$ for any $m \in \mathbb{N}$, so diam $\bigcap_{n=1}^{\infty} V_n \leqslant$ diam V_m. Hence diam $\left(\bigcap_{n=1}^{\infty} V_n\right) \leqslant \inf\{\text{diam } V_m : m \in \mathbb{N}\} = m_0$ say.

Conversely, m_0 is a lower bound for the diameters of the V_n, so for any $\varepsilon > 0$ and any $n \in \mathbb{N}$ we know that diam $V_n > m_0 - \varepsilon$. Hence there exist points $x_n, y_n \in V_n$ such that $d(x_m, x_n) > m_0 - \varepsilon$. Since X is sequentially compact, (x_n) has a subsequence (x_{n_r}) converging to a point $x \in X$, and then (y_{n_r}) has a subsequence $(y_{n_{r_s}})$ converging to a point $y \in X$. Since $(x_{n_{r_s}})$ is a subsequence of (x_{n_r}) it too converges to x. Also, by continuity of the metric, $d(x_{n_{r_s}}, y_{n_{r_s}}) \to d(x, y)$ as $s \to \infty$. Hence $d(x, y) \geqslant m_0 - \varepsilon$. Also, $x, y \in V_n$ for each $n \in \mathbb{N}$ since V_n is closed in X. Since this is true for all $n \in \mathbb{N}$, we have $x, y \in \bigcap_{n=1}^{\infty} V_n$. Hence diam $\bigcap_{n=1}^{\infty} V_n \geqslant m_0 - \varepsilon$. But this is true for any $\varepsilon > 0$, so diam $\bigcap_{n=1}^{\infty} V_n \geqslant m_0$.

The above taken together prove the result.

14.16(a) Any element of $\bigcap_{n=1}^{\infty} V_n$ must be in V_1, so it is the function f_m for some $m \in \mathbb{N}$. But $f_m \notin V_n$ for $n > m$. So (a) holds.

(b) For any two distinct elements f_l, f_m of V_n we know that $d_\infty(f_l, f_m) = 1$. This shows that diam $V_n = 1$.

(c) In this case, diam $\bigcap_{n=1}^{\infty} V_n = 0$, but $\inf\{\text{diam} V_n : n \in \mathbb{N}\} = 1$. So the conclusion of Exercise 14.15 fails. (We note that the space $\{f_n : n \in \mathbb{N}\}$ with the sup metric is not compact - see Example 14.23.)

14.17 (a) Let $x \in X$. We want to show that $x \in f(X)$. Consider the sequence (x_n) in X defined by:

$$x_1 = x, \quad x_{n+1} = f(x_n) \quad \text{for all integers } n \geqslant 1.$$

Since X is sequentially compact, there is a convergent subsequence, say (x_{n_r}). Any convergent sequence is Cauchy, so given $\varepsilon > 0$ there exists $R \in \mathbb{N}$ such that $|x_{n_r} - x_{n_s}| < \varepsilon$ whenever $s > r \geqslant R$, in particular $|x_{n_R} - x_{n_r}|$ whenever $r > R$. Now we use the isometry condition, iterated $n_R - 1$ times, to see that $|x_1 - x_{n_r - n_R + 1}| < \varepsilon$ whenever $r \geqslant R$. But $x_1 = x$ and $x_{n_r - n_R + 1} \in f(X)$ whenever $r > R$. Hence $x \in \overline{f(X)}$. But X is compact and f is continuous, so $f(X)$ is compact. Also, X is metric hence Hausdorff, so $f(X)$ is closed in X. Hence $\overline{f(X)} = f(X)$. So $x \in f(X)$ for any $x \in X$, which says that f is onto. Hence f is an isometry.

(b) We can apply (a) to the compositions $g \circ f : X \to X$ and $f \circ g : Y \to Y$ to see that these are both onto. Since $g \circ f$ is onto, g is onto. Similarly since $f \circ g$ is onto, f is onto. Hence both f and g are isometries.

(c) We just define $f : (0, \infty) \to (0, \infty)$ by $f(x) = x + 1$.