Limit Point Compactness

Here are detailed proofs of two results:

THEOREM 1. If a topological space X is compact, then it is limit point compact (every infinite subset has a limit point).

PROOF Suppose this is false, and let $A \subseteq X$ be an infinite set with no limit points. Since $L(A) = \phi \subseteq A$, we know A is closed. Let ~~$a \in A$. Then $a \notin L(A)$, so there is some open neighborhood~~ $C(a) = A - \{a\}$ for $a \in A$. Then $L(C(a)) = \phi$ $\Longrightarrow$ $C(a)$ is also closed in X. Now the family $\{C(a)\}$ has the finite intersection property because $\cap\, C(a_i) = A - \{a_1, \ldots, a_n\}$ is the complement of a finite set in the infinite set A. However, $\bigcap_{a \in A} C(a) = \phi$, which implies that X cannot be compact. ∎

THEOREM 2. If X is a metric space, then X is sequentially compact $\Longleftrightarrow$ X is limit point compact.

PROOF. 1 Suppose X is sequentially compact; we want to show X is limit point compact.

Let $A \subseteq X$ be an infinite subset, and pick out distinct points $a_1, a_2, \ldots$ in X. Passing to a subsequence if necessary, we may assume $\{a_n\}$ converges (by sequential compactness). If $b = \lim a_n$, then $b \in L(A)$ by general considerations involving limit points and limits of sequences. ∎

2 Now suppose X is limit point compact, and let $\{a_n\}$ be a sequence. If ∞ many $a_n =$ some c, then the sequence has a convergent subsequence, so we might as well assume a_n has ∞ many values. Also, we might as well assume $a_1, \ldots, a_n$ are distinct by passing to a subsequence if necessary.

Since X is limit point compact, there is some $b \in L(a_1, a_2, \ldots)$. Therefore for each $k > 0$ there is some a_{n_k} such that $n_k > n_{k-1} > \ldots$ and $d(b, a_{n_k}) < 1/k$.

It follows that $b = \lim_{k \to \infty} a_{n_k}$, and therefore X is sequentially compact. ∎