SOME NOTES ON LIMIT POINT COMPACTNESS

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1. Limit Point Compactness

This is a topic I haven't seen covered in my courses, but you're being exposed to it. The definition goes something like:

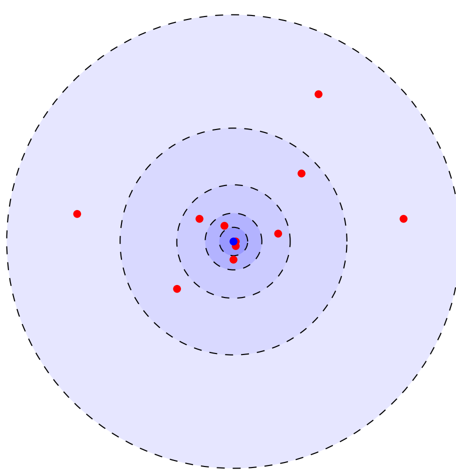
Definition 1. A topological space X is **limit-point compact** if every infinite subset $S \subseteq X$ contains a limit point in X.

This isn't terribly enlightening, but let's consider a "practical" approach to the statement. For this, we need to return to another definition.

Definition 2. Suppose we have a subset $S \subseteq X$ in a topological space X. We say a point $x \in X$ is a **limit point of** $\boldsymbol{S}$ if, for any punctured neighborhood $U_x - x$ of x,

$$(U_x - x) \cap S \neq \emptyset.$$

As is common, we should think of something like $\mathbb{R}^2$, with the usual metric topology. In that case, we have that no matter how small the radius r, $B_x(r) - x$ contains *some* point in S. In a picture, with red dots as elements in S and the blue dot as the limit point x, we have



Note that I'm only showing a few radii.

Now, we often talk of a point containing all its limit points as being closed. In the sense of a closed interval in $\mathbb{R}$, that should be pretty clear. However, the proof today was a bit different.

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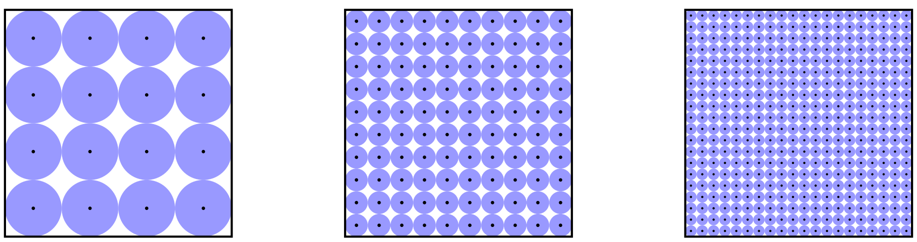
Here, we are proving the set contains ALL its limit points vacuously, because there are none! What does this even look like?

Well, a single point $\{a\}$ in $\mathbb{R}$ satisfies this requirement. It has no limit point, because any punctured neighborhood centered at a doesn't contain any point of the set (which only contains a). Also, any point $x \neq a$ lies a distance $|x-a|$ away, so a punctured neighborhood of lesser radius (say, $r = |x-a|/2$) will also not contain a. This same argument applies to any subset of $\mathbb{R}$ that doesn't contain an interval. This is another way to see that $\mathbb{Z}$ is a closed subset of $\mathbb{R}$, even if it isn't compact.

Now, what does all of this have to do with the idea behind limit-point compactness? If we have an infinite subset S of our space X, and it has no limit points, it means that around each point in S, there is some punctured neighborhood (think disk) that doesn't touch any element in S. If X is of limited "size" (think of area), then we couldn't fit an infinite number of neighborhoods/disks within it. Some surrounding neighborhood has to crush down. As a result, there must be *some* limit point of the set S somewhere in X, i.e., one point where an arbitrarily small neighborhood always contains a point of S.



Can I place an infinite number of fixed radius neighborhoods inside a compact space?

In a non-metric topological space, the basic concept is the same, replacing disks with open neighborhoods - it's just harder to draw.

Finally, I mention for use in the proof,

Definition 3. A topological space X is **compact** if any open cover $\bigcup_{\alpha \in A} U_\alpha$ admits a finite subcover $\bigcup_{i=1}^{n} U_{\alpha_i}$.

2. A Related Proof

Theorem. *If a topological space X is compact, then it is limit-point compact.*

Proof. This proof requires you to know and use the definition of both types of compactness, the often mentioned finite intersection property, as well as the rule that a set which contains all its limit points is closed. The only things we have to work with by reading the theorem are the two definitions. As Dr. Schultz mentioned in class, there isn't an obvious way to prove it directly, so we prove it instead by contradiction. Our goal is to get a contradiction related to X being compact. Again, compactness means *any* open cover of X admits a finite subcover.

Suppose X is compact, but it is not limit-point compact. Just negating the definition of LPC, there must exist some infinite subset $S \subseteq X$ that has no limit points in X, so the set of limit points is empty ($L(A) = \emptyset$). But then

$$\emptyset \;=\; L(A) \;\subseteq\; A,$$

so A contains all its limit points, and therefore is closed. We want to get to a collection of open sets (a cover of X) from a closed set. We can do that by considering

$$C(a) \;=\; A - \{a\},$$

for each $a \in A$. Notice that $\emptyset = L\left(C(a)\right) \subseteq C(a)$, so $C(a)$ is indeed closed. Another intuitive way to think of it is, we assumed A didn't have any limit points, so each point in A can be contained in a neighborhood that doesn't contain any other points of A. Removing one point from the whole set won't change that.

Now, if take a finite collection of points $\{a_i\}_{i=1}^n \subseteq A$, and then consider

$$\bigcap_{i=1}^{n} A - C(a_i) \;=\; A - \bigcup_{i=1}^{n} C(a_i),$$

it should be clear this is nonempty, satisfying the finite intersection property. On the other hand

$$\bigcap_{a \in A} A - C(a) \;=\; \emptyset,$$

as each $C(a)$ is actually missing exactly one element of A. Let's call $A - C(a) = F(a)$ to save a bit of writing, noting that $F(a)$ is a closed set. Moroever, $X - F(a)$ is the open set comprised of the complement of A in X, plus the complement of $\{a\}$ in A. This is open (as the complement of a closed set), and

$$\bigcup_{a \in A} (X - F(a)) \;=\; X$$

is an open cover. If this were compact, we could find some subcover such that

$$\bigcup_{i=1}^{n} (X - F(a_i)) \;=\; X - \left(A - \bigcup_{i=1}^{n} C(a_i) \right) \;=\; X,$$

a contradiction. Since we know that

$$A - \bigcup_{i=1}^{n} C(a_i) \;=\; \bigcap_{i=1}^{n} A - C(a_i) \;\neq\; \emptyset,$$

we know

$$X - \left(A - \bigcup_{i=1}^{n} C(a_i) \right) \;\neq\; X,$$

as <u>*something*</u> is being removed. This is a contradiction, as we have found an open cover of X that doesn't admit a finite subcover. Hence, our assumption that X is not limit-point compact was false, and the theorem is proved. □