I.3 Completions

Theorem I.3.1 Let (X, d) be a metric space. Then there is a complete metric space (X^*, d^*) and an isometry $f : X \to X^*$ such that $\overline{f[X]} = X^*$.

into, maybe not onto

We shall say that $(X^*, d^*; f)$ is a *completion* of (X, d).

Theorem I.3.2 (Uniqueness of completions) If $(X_1, d_1; f_1)$ and $(X_2, d_2; f_2)$ are completions of (X, d), then there is a unique isometry $\varphi : (X_1, d_1) \longrightarrow (X_2, d_2)$ such that $f_2 = \varphi \circ f_1$.

$$\begin{tikzcd} & (X_1, d_1) \arrow[dd, "\varphi"] \\ (X, d) \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\ & (X_2, d_2) \end{tikzcd}$$

Proof of I.3.1 (Existence)

Fix $a \in X$, and define $f_0 : X \longrightarrow BC(X)$ by $f_0(x)(y) = d(y,x) - d(y,a)$. Need to show $f_0(x)$ is continuous and lies in $BC(X)$.

<u>Bounded</u>: $|f_0(x)(y)| \leq |d(y,x) - d(y,a)| \leq d(x,a)$

In fact, it takes a maximum value if $y = x$ or a, and this is just $d(x,a)$.

<u>Continuous</u>: It will suffice to show that f_0 is distance preserving (hence isometry, hence uniformly continuous).

$$d(f_0(x), f_0(y)) = \underset{z}{\text{l.u.b.}} \left| (d(z,x) - d(z,a)) - (d(z,y) - d(z,a)) \right| =$$

$$\underset{z}{\text{l.u.b.}} \left| d(z,x) - d(z,y) \right| \leq d(x,y),$$

and the value on the right is attained at $z = x$ or y. Hence f_0 is distance preserving.

To finish the proof, let $X^* = \overline{f[X]}$. ∎

closed in a complete metric space

Proof of I.3.3 (Uniqueness)

Let (X^*, d^*) be the example constructed in the proof of I.3.2. For $i = 1, 2$ we shall construct a unique isometry $\varphi_i : X_i \longrightarrow X^*$ such that $f^* \circ \varphi_i = f_i$.

This suffices, for $\varphi_2^{-1} \circ \varphi_1 = \varphi$ will be an isometry with $f_2 = f_1 \circ \varphi$. Moreover, φ is unique; if $f_2 = f_1 \circ \varphi'$ also holds for some φ', then $\varphi = \varphi'$ on the dense subset $f_1[X]$, so they agree everywhere.

Consider the following "commutative diagram" (both composites agree).

$$\begin{tikzcd}
X_i \arrow[r, "e_i"] & BC(X_i) \arrow[d, "J_i"] \\
X \arrow[u, "f_i"] \arrow[r, "e"] & BC(X)
\end{tikzcd}$$

J_i restriction map

e = previously constructed map, e_i = its analog.

$e = J_i \circ e_i \circ f_i$

We then have $\overline{e[X]} = \overline{J_i e_i f_i[X]} \supseteq$ $J_i e_i[X_i]$ by continuity.

RECALL: $g : Y \to Z$ continuous $\iff$ for all $A \subseteq Y$, $g[\overline{A}] \subseteq \overline{g[A]}$.

CLAIM $J_i e_i$ is distance preserving.

Let $h_i : X_i \times X_i \longrightarrow \mathbb{R}$ be defined by

$$h_i(u, v) = d(J_i e_i(u), J_i e_i(v)).$$

We need to show $h_i = d_i$. Since $f_i[X] \times f_i[X]$ is dense in $X_i \times X_i$, we need only check this on the smaller set. But here we have

RECALL: $A \subseteq Y$, $B \subseteq Z \Rightarrow$ $\overline{A \times B} = \overline{A} \times \overline{B}$

$$h_i(f_i(r), f_i(s)) = d(J_i e_i f_i(r), J_i e_i f_i(s))$$

$$= d(e(r), e(s)) \underset{\text{I.3.1}}{\overset{\text{by}}{=\!=}} d(r, s) = d_i(f_i(r), f_i(s)),$$

proving the claim.

Conclusion of the proof of I.3.2 Since $J_i e_i$ is distance preserving, its image is a complete, hence closed, subset of $BC(X)$

containing $e[X]$ (because $e = J_i e_i f_i$). Therefore $J_i e_i[X_i] \supseteq \overline{e[X]}$, and by the previous discussion they are equal. But this means that $J_i e_i$ defines an isometry φ_i from X_i onto $\overline{e[X]} = X^*$. The uniqueness of φ_i follows as before; if $\varphi_i' f_i = \varphi_i f_i$, then $\varphi_i' = \varphi_i$ on the dense subset $\overline{f_i[X]}$, so they must agree everywhere. ∎