

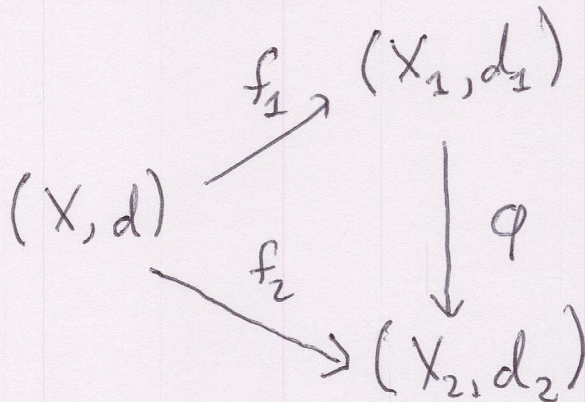
I.3 Completions

Theorem I.3.1 Let (X, d) be a metric space. Then there is a complete metric space (X^*, d^*) and an isometry $f: X \rightarrow X^*$ such that $\overline{f[X]} = X^*$. into, maybe not onto

We shall say that $(X^*, d^*; f)$ is a completion of (X, d) .

Theorem I.3.2. (Uniqueness of completions)

If $(X_1, d_1; f_1)$ and $(X_2, d_2; f_2)$ are completions of (X, d) , then there is a unique isometry $\varphi: (X_1, d_1) \rightarrow (X_2, d_2)$ such that $f_2 = \varphi \circ f_1$.



PROOF OF I.3.1 (Existence)

Fix $a \in X$, and define $f_0: X \rightarrow BC(X)$ by $f_0(x)(y) = d(y,x) - d(y,a)$. Need to show $f_0(x)$ is continuous and lies in $BC(X)$.

Bounded: $|f_0(x)(y)| \leq |d(y,x) - d(y,a)| \leq d(x,a)$
 In fact, it takes a maximum value if $y = x$ or a , and this is just $d(x,a)$.

Continuous: It will suffice to show that

f_0 is distance preserving (hence isometry, hence uniformly continuous).

$$d(f_0(x), f_0(y)) = \text{l.u.b.}_z \left| (d(z,x) - d(z,a)) - (d(z,y) - d(z,a)) \right| =$$

$$\text{l.u.b.}_z |d(z,x) - d(z,y)| \leq d(x,y),$$

and the value on the right is attained at $z = x$ or y . Hence f_0 is distance preserving.

To finish the proof, let $X^* = \overline{f[X]}$. \blacksquare

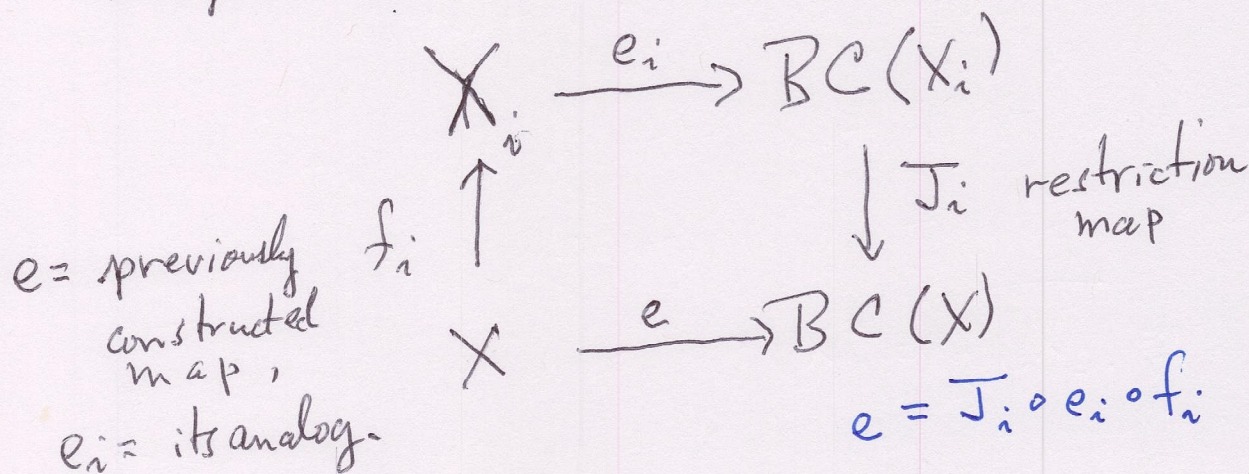
closed in a complete metric space

PROOF OF I.3.3 (Uniqueness)

Let (X^*, d^*) be the example constructed in the proof of I.3.2. For $i=1, 2$ we shall construct a unique isometry $\varphi_i: X_i \rightarrow X^*$ such that $f^* \circ \varphi_i = f_i$.

This suffices, for $\varphi_2^{-1} \circ \varphi_1 = \varphi$ will be an isometry with $f_2 = f_1 \circ \varphi$. Moreover, φ is unique; if $f_2 = f_1 \circ \varphi'$ also holds for some φ' , then $\varphi = \varphi'$ on the dense subset $f_1[X]$, so they agree everywhere.

Consider the following "commutative diagram" (both composites agree).



We then have $\overline{e[X]} = \overline{\bigcup_i e_i f_i[X]} \supseteq$
 $\bigcup_i \overline{e_i[X_i]}$ by continuity.

RECALL: $g: Y \rightarrow Z$
 continuous \Leftrightarrow for all
 $A \subseteq Y, g[A] \subseteq \overline{g[A]}$.

CLAIM $\bigcup_i e_i$ is distance preserving.

Let $h_i: X_i \times X_i \rightarrow \mathbb{R}$ be defined by

$$h_i(u, v) = d(\bigcup_i e_i(u), \bigcup_i e_i(v)).$$

We need to show $h_i = d_i$. Since $f_i[X] \times f_i[X]$ is dense in $X_i \times X_i$, we need only check this on the smaller set. But here we have

$$\begin{aligned} h_i(f_i(r), f_i(s)) &= d(\bigcup_i e_i f_i(r), \bigcup_i e_i f_i(s)) \\ &= d(e(r), e(s)) \stackrel{\text{by I.3.1}}{=} d(r, s) = d_i(f_i(r), f_i(s)), \end{aligned}$$

RECALL:
 $A \subseteq Y, B \subseteq Z \Rightarrow$
 $\overline{A \times B} = \overline{A} \times \overline{B}$

proving the claim.

Conclusion of the proof of I.3.2 Since

$\bigcup_i e_i$ is distance preserving, its image is a complete, hence closed, subset of $BC(X)$

I.3.5

containing $e[X]$ (because $e = \sum_i e_i f_i$).

Therefore $\sum_i e_i [X_i] \supseteq \overline{e[X]}$, and by the previous discussion they are equal.

But this means that $\sum_i e_i$ defines an isometry φ_i from X_i onto $\overline{e[X]} = X^*$. The

uniqueness of φ_i follows as before; if

$\varphi'_i f_i = \varphi_i f_i$, then $\varphi'_i = \varphi_i$ on the dense

subset $\overline{f_i[X]}$, so they must agree everywhere. \square