SOLUTIONS TO EXERCISES FOR

MATHEMATICS 145B — Part 1

Spring 2015

0.6 : Components

Additional exercises

1. (i) First of all, if C and D are connected subsets of X and Y respectively then $C \times D$ is connected, so $C \times D$ is contained in a component K. We shall prove that $C \times D = K$. Let p_X and p_Y be the projection maps onto the X and Y coordinates. Then the sets $p_X[K]$ and $p_Y[K]$ are connected subsets of X and Y respectively, and since $C \times D \subset Y$ we know that $C \subset p_X[K]$ and $D \subset p_Y[K]$. However, we also know that C and D are maximal connected subsets of X and Y, and therefore it follows that $C = p_X[K]$ and $D = p_Y[K]$. The latter equations imply that $K \subset C \times D$, and since we already have the reverse inclusion this yields the desired identity $K = C \times D$.■

(ii) Exactly the same argument goes through with "arc component" replacing "connected component" and "arcwise connected" replacing "connected." The only properties of connectedness we are using are that a product of connected spaces is connected, a continuous image of a connected set is connected, and a connected component is a maximal connected subset. Each of these statements has a valid analog for arcwise connected spaces.■

2. Let $\mathcal{R}$ be the equivalence relation, let $x \in X$, and suppose that C_x is the equivalence class of x. The assumption means that if $y \in C_x$, then there is an open subset U_y such that $y \in U_y$ and $U_y \subset C_x$. Therefore we have

$$C_x \quad = \quad \bigcup_{y \in C_x} \{y\} \quad \subset \quad \bigcup_{y \in C_x} U_y \quad \subset \quad C_x$$

so that C_x is the union of the open sets U_y and hence is open in X.

Now the complement $X - C_x$ is a union of all the remaining equivalence classes, and therefore $X - C_x$ is also open. Therefore we have a decomposition of X into two disjoint open subsets, and one of them (namely, C_x) is nonempty. Since X is connected, the other subset must be empty. But this means that $X = C_x$ and hence X is connected.■

3. We shall use the preceding exercise. Let $\mathcal{A}$ be the equivalence relation whose equivalence classes are the arc components of U, let $x \in U$, and let C_x denote the arc component of U containing x. Assume further that U is connected. Given $y \in A_x \subset U$ choose $r(y) > 0$ so that the open disk neighborhood $N_{r(y)}(y; \mathbb{R}^n)$ is contained in U. This open disk neighborhood is arcwise connected, and since $y \in A_x$ it follows that $N_{r(y)}(y; \mathbb{R}^n) \subset A_x$. Therefore the equivalence relation $\mathcal{A}$ satisfies the condition in the preceding exercise, and by the conclusion of the latter we can conclude that $\mathcal{A}$ has one equivalence class, which means that U is arcwise connected.■

4. Let A denote the set described in the exercise, let C denote the unit circle defined by $x^2 + y^2 = 1$, and consider the continuous mapping $f : C \times [1, \sqrt{2}] \to A$ which sends (z, c) to cz. We know that C is connected because it is the continuous image of the interval $[0, 1]$ under the mapping $\gamma(t) = (\cos 2\pi t, \sin 2\pi t)$, and therefore the product is connected. The mapping f is a homeomorphism with an explicit inverse given by sending v to $(|v|^{-1}\, v, |v|)$, and therefore it follows that A must also be arcwise connected.

5. Follow the hint. If $x, y \in \mathbb{R}^n - \{\mathbf{0}\}$ are linearly independent, then the line segment $ty + (1-t)x$ $(0 \leq t \leq 1)$ does not pass through $\mathbf{0}$ (if it did, then this would occur for some value of t such that $0 < t < 1$, so that $ty = ((t-1)x$ and the vectors x and y would be linearly dependent), and hence x and y lie in the same arc component of $\mathbb{R}^n - \{\mathbf{0}\}$. Suppose now that x and y are linearly dependent, so that each is a nonzero multiple of the other. Since $n \geq 2$ there is some vector z such that x and z are linearly independent, and it follows immediately that y and z are also linearly independent. Two applications of the previous argument then show that x, y, z all lie in the same arc component. Thus in all cases we have shown that two arbitrary points $x, y \in \mathbb{R}^n - \{\mathbf{0}\}$ always lie in the same arc component, which means that this space is arcwise connected.

To prove the result for the sphere, it suffices to note that S^{n-1} is the continuous image of $\mathbb{R}^n - \{\mathbf{0}\}$ under the mapping sending v to $|v|^{-1} \cdot v$.■

I. Complete metric spaces

I.1 : Definitions and Basic Properties

Problems from Munkres, § 43, p. 270 – 271

1. (a) Let $\{ x_n \}$ be a Cauchy sequence in X and choose M so large that $m, n \geq M$ implies $\mathbf{d}(x_m, x_n) < \varepsilon$. Then all of the terms of the Cauchy sequence except perhaps the first $M - 1$ lie in the closure of $N_\varepsilon(x_M)$, which is compact. Therefore it follows that the sequence has a convergent subsequence $\{ x_{n(k)} \}$. Let y be the limit of this subsequence; we need to show that y is the limit of the entire sequence.

Let $\eta > 0$ be arbitrary, and choose $N_1 \geq M$ such that $m, n \geq N_1$ implies $\mathbf{d}(x_m, x_n) < \eta/2$. Similarly, let $N_2 \geq M$ be such that $n(k) > N_2$ implies $\mathbf{d}(x_{n(k)}, y) < \eta/2$. If we take N to be the larger of N_1 and N_2, and application of the Triangle Inequality shows that $n \leq N$ implies $\mathbf{d}(x_n, y) < \eta$. Therefore y is the limit of the given Cauchy sequence and X is complete.■

(b) Take $U \subset \mathbb{R}^2$ to be the set of all points such that $xy < 1$. This is the region "inside" the hyperbolas $y = \pm 1/x$ that contains the origin. It is not closed in $\mathbb{R}^2$ and therefore cannot be complete. However, it is open and just like all open subsets U of $\mathbb{R}^2$ if $x \in X$ and $N_\varepsilon(x) \subset U$ then $N_{\varepsilon/2}(x)$ has compact closure in U.■

4. There are two parts to the proof:

(i) Prove that the intersection $\cap_n A_n$ is nonempty.

(ii) Prove that the intersection $\cap_n A_n$ contains no more than one point.

We shall begin by proving the first statement. Choose a sequence of points a_n such that $a_n \in A_n$ for each n. CLAIM: This is a Cauchy sequence. — Let $\varepsilon > 0$ and choose N such that $k \geq N$ implies $\mathrm{diam}(A_k) < \varepsilon$. Suppose now that $m, n \geq N$. Since $k \geq N$ implies $A_k \subset A_N$, we have $a_m, a_n \in A_N$, so that $d(a_m, a_n) < \varepsilon$ and hence $\{a_n\}$ is a Cauchy sequence. Since the underlying metric space is complete, this sequence has a limit which we shall call a.

To prove that the intersection is nonempty, it will suffice to show that $a \in \cap_n A_n$. As before, we know that $k \geq n$ implies that $A_k \subset A_n$, and therefore all but finitely many points in the sequence $\{a_k\}$ lie in A_n. Since

$$\lim_{k \to \infty} a_k \quad = \quad \lim_{j \to \infty} a_{j+n}$$

it follows that a is the limit of the second sequence; the values of the latter lie in A_n, and since A_n is closed in X it follows that the limit value a also lies in A_n. Since n was arbitrary, it follows that $a \in \cap_n A_n$.

To conclude the proof, we must show that the intersection contains at most one point. Suppose that $u, v \in \cap_n A_n$. Then $u, v \in A_n$ implies that $d(u,v) \leq \mathrm{diam}(A_n)$ for all n and hence that

$$0 \quad \leq \quad d(u,v) \quad \leq \quad \lim_{n \to \infty} \mathrm{diam}(A_n) \quad = \quad 0$$

which means that $d(u,v) = 0$ and hence $u = v$.■

6. (b) Suppose that X and Y have complete metrics d_X and d_Y. Consider the metric on $X \times Y$ defined by

$$d_\infty\big((x_1,y_1),(x_2,y_2) \big) \quad = \quad \max\big(d_X(x_1,x_2),\, d_Y(y_1,y_2) \big) \ .$$

With respect to this metric, a sequence $\{(p_n, q_n)\}$ has a limit if and only if the coordinates do, and in this case we have $(p_n, q_n) \to (p, q)$ where $x_n \to p$ and $y_n \to q$. This is true because $d_\infty \geq d_X, d_Y$ and $d_\infty \leq d_X + d_Y$.

Assuming that X and Y are complete, suppose that $\{(p_n, q_n)\}$ is a Cauchy sequence in the product. Then the inequality at the end of the previous paragraph implies that both $\{p_n\}$ and $\{q_n\}$ are also Cauchy sequences, so that these sequences have limits p and q respectively. By the final statement in the preceding paragraph, it follows that $(p_n, q_n) \to (p, q)$, and therefore $X \times Y$ is complete with respect to the metric d_∞.■

(c) We shall follow the hint in Munkres. By (b) we know that $X \times \mathbb{R}$ is complete in the d_∞ metric, so it is only necessary to check that the graph of the function

$$\varphi(x) \quad = \quad \frac{1}{d(x, X - U)}$$

is a closed subset of $X \times \mathbb{R}$ (since φ is a continuous function we know that U is homeomorphic to the graph of φ); we should stress that we want to show that the graph is closed in $X \times \mathbb{R}$, and it is not enough to show that it is closed in $U \times \mathbb{R}$ (this is true by continuity and the fact that all spaces under consideration are Hausdorff).

Here is the proof that the graph of φ is closed in $X \times \mathbb{R}$: Suppose that we have a sequence $(x_n, \varphi(x_n))$ in $U \times \mathbb{R}$ which converges to a point $(y, s) \in X \times \mathbb{R}$; we need to show that $y \in U$ and $s = \varphi(y)$. As in part (b), we know that $x_n \to y$ and $\varphi(x_n) \to s$. If $y \in U$ then by the continuity of φ we know that $s = \varphi(y)$, so it is only necessary to eliminate the possibility that $y \notin U$. In this case we have $d(y, X - U) = 0$ because $X - U$ is closed in X and we also have $d(x_n, X - U) \to d(y, U) = 0$. But the latter means that $\varphi(x_n) \to +\infty$ as $n \to \infty$, which contradicts our assumption that $\varphi(x_n) \to s \in \mathbb{R}$. The source of his contradiction was our assumption that $y \notin U$, so this must be false. By the previous reasoning it follows that (y, s) lies in the graph of U and hence the latter is a closed subset of the complete metric space $X \times \mathbb{R}$.■

Note. For the sake of completeness, here is a proof that the graph G of a continuous function $f : X \to Y$ is homeomorphic to X. The graph is the image of the function $F : X \to X \times Y$

defined by $F(x) = (f(x), g(x))$; let $F_0 : X \to G$ be the induced continuous mapping from X to $G \subset X \times Y$. This map is continuous and onto, and it is also 1–1 because $F(x) = F(x')$ implies the first coordinates of the latter are equal; since these first coordinates are x and x', it follows that $F(x) = F(x')$ implies $x = x'$. Furthermore, the inverse to F_0 is simply the coordinate projection sending (x, y) to x, so F_0 also has a continuous inverse, and therefore F_0 is a homeomorphism.∎

Additional exercises

1. Suppose that X is a discrete metric space and $\{a_n\}$ is a Cauchy sequence in X. Let N be such that $m, n \geq N$ implies $d(a_m, a_n) < 1$. Since the neighborhoods of the form $N_1(p)$ are one point sets for all $p \in X$, it follows that $a_m = a_n$ for $m, n \geq N$; denote this common value by L. We then have that $k \geq N$ implies $d(a_k, L) = 0$, so the sequence converges very strongly — for all but finitely many n we have $a_n = L$.∎

2. Follow the hint, and try to see what a function in the intersection would look like. In the first place it has to satisfy $f(0) = 1$, but for each $n > 0$ it must be zero for $t \geq 1/n$. The latter means that the $f(t) = 0$ for all $t > 0$. Thus we have determined the values of f everywhere, but the function we obtained is not continuous at zero. Therefore the intersection is empty. Since every function in the set A_n takes values in the closed unit interval, it follows that if f and g belong to A_n then $\|f - g\| \leq 1$ and thus the diameter of A_n is at most 1 for all n. In fact, the diameter is exactly 1 because $f(0) = 1$.

For the sake of completeness, we should note that each set A_n is nonempty. One can construct a "piecewise linear" function in the set that is zero for $t \geq 1/n$ and decreases linearly from the 1 to 0 as t increases from 0 to $1/n$. (Try to draw a picture of the graph of this function!)∎

3. We start by justifying the assertion in the hint. If we divide both sides of the inequality $m \cdot d'(x, y) \leq d(x, y)$ by m we obtain the inequality $\cdot d'(x, y) \leq m^{-1} d(x, y)$, and if we divide both sides of the inequality $d(x, y) \leq M \cdot d'(x, y)$ by M we obtain the inequality $M^{-1} \cdot d(x, y) \leq d'(x, y)$. If we combine these we obtain

$$M^{-1} \cdot d(x, y) \quad \leq \quad d'(x, y) \quad \leq \quad m^{-1} \cdot d(x, y) \ .$$

This means that the assumptions in the exercise turn out to be symmetric with respect to d and d', which implies that we only need to show one direction of the implication; namely, if the metric is complete with respect to d, then it is also complete with respect to d'.

Suppose that $\{x_n\}$ is a Cauchy sequence with respect to d'. Let $\varepsilon > 0$, and choose N so that $p, q \geq N$ implies $d'(x_p, x_q) < M^{-1}\varepsilon$. Then we also have

$$d(x_p, x_q) \quad \leq \quad M\, d'(x_p, x_q) \quad < \quad M\, M^{-1}\varepsilon \quad = \quad \varepsilon$$

which means that $\{x_n\}$ is also a Cauchy sequence with respect to d. Therefore the sequence has some limit L with respect to d. We claim that L is also the limit of the sequence with respect to d'.

Once again let $\varepsilon > 0$. Since $x_n \to x$ with respect to d, there is some K such that $k \geq K$ implies $d(x_k, L) < m\,\varepsilon$. As before, we now also have

$$d'(x_k, L) \quad \leq \quad m^{-1}\, d(x_k, L) \quad < \quad m^{-1} m\, \varepsilon \quad = \quad \varepsilon$$

which means that $\{x_k\}$ is also converges to L with respect to d', and consequently X is also complete with respect to d'.∎

I.2 : The Contraction Lemma

Additional exercises

1. Follow the hint. The fact that x is a root of the polynomial equation if and only if $f(x) = x$ follows by adding $1 - x^5$ to both sides and then dividing both sides by 7. By construction $f(0) = \frac{1}{7}$, it is decreasing on $[0,1]$ because $f'(x) = -\frac{5}{7}\,x^4 \le 0$ on that interval, and $f(0) = 0$. These combine to show that f maps $[0,1]$ to itself. By the Mean Value Theorem, the hypothesis of the Contraction Lemma will be satisfied if $|f'| \le \alpha < 1$ for some α satsifying $0 < \alpha < 1$, and the formula for f' shows that we can take $\alpha = \frac{5}{7}$. Therefore by the Contraction Lemma there is a unique $a \in [0,1]$ such that $f(a) = a$, or equivalently $a^5 + 7a - 1 = 0$. Since $f(0) \ne 0$ and $f(1) \ne 1$, it follows that $0 < a < 1$.∎

2. Let $X = \mathbb{R}$ and let $f(x) = x + 1$. Then $|f(x) - f(y)| = |x - y|$ but $f(x) \ne x$ for all x.∎

3. If $C < 1$ then the conclusion is an immediate consequence of the Contraction Lemma. Suppose now that $C > 1$. Since f is 1–1 onto, it has an inverse which we shall call g. Since f and g are inverses, we know that $f \circ g$ and $g \circ f$ are the identity mappings, and therefore we have

$$d(x,y) \quad = \quad d\big(\, f(g(x)), f(g(y))\,\big) \quad = \quad C \cdot d\big(\, g(x), g(y)\,\big)$$

or equivalently $d\big(\, g(x), g(y)\,\big) = C^{-1}\, d(x,y)$, where $C^{-1} < 1$. Therefore by the Contraction Lemma there is a unique point $p \in X$ such that $g(x) = x$.

We shall conclude the argument by showing that $g(x) = x$ if and only if $f(x) = x$. This is true because $g(x) = x$ implies $x = f(g(x)) = f(x)$ and similarly $f(x) = x$ implies $x = g(f(x)) = g(x)$.∎

I.3 : Completions

Additional exercises

1. Let X^* denote the completion of X. Then a subset $C \subset X$ is complete if and only if it is closed in X^*. Therefore we may prove the assertions in the exercise as follows:

(1) If $C_1, C_2 \subset X$ are complete, then they are closed in X^* and hence their union is also closed in X^*. But this means that their union is complete.

(2) If $C_\alpha \subset X$ is complete for each α, then each such set is closed in X^* and hence their intersection is also closed in X^*. But this means that the intersection is also complete.∎

2. In order to show that A is dense in Y it suffices to show that if $y \in Y$ and $\varepsilon > 0$, then there is some $a \in A$ such that $d(y,a) < \varepsilon$. We know that X is dense in Y, and therefore if $\varepsilon > 0$ there is some $x \in X$ such that $d(x,y) < \frac{1}{2}\,\varepsilon$, and likewise we know that A is dense in X, so that there is some $a \in A$ such that $d(a,x) < \frac{1}{2}\,\varepsilon$. By the Triangle Inequality we then have $d(a,y) < \varepsilon$, and therefore A is dense in Y. This means that Y is also a completion of A.∎