Familiar gluing constructions

The purpose of this document is to give a detailed mathematical interpretation of two familiar constructions of 3-dimensional geometrical figures from 2-dimensional ones by gluing parts of their edges together. The first of these is the construction for the lateral surface of a cone from a solid circular sector as on the next page, and the idea is to cut out the sector colored in green and to glue the two radial edges together in the standard way.

In order to see that the quotient space defined in the preceding paragraph is topologically equivalent to a cone we have to do the following:

(1) Formally describe the equivalence relation $\mathcal{R}$ on the planar sector by the indicated gluing construction.

(2) Construct an explicit lateral surface of a cone in $\mathbb{R}^3$.

(3) Construct a map φ from the quotient space onto the subset in (2) such that two points in the planar sector have the same image under φ if and only if they are $\mathcal{R}$-equivalent.

Formally, let α be the central angle for the green circular sector, and let p be the radius of the circle. The first order of business is to find the radius q for the base of the associated cone and the altitude h of this cone. Now the arc corresponding to the larger sector corresponds to the boundary circle for the base of the cone, so the circumference of this circle is $p(2\pi - \alpha)$. Therefore the radius q of this circle is equal to

$$\frac{p(2\pi - \alpha)}{2\pi} \; .$$

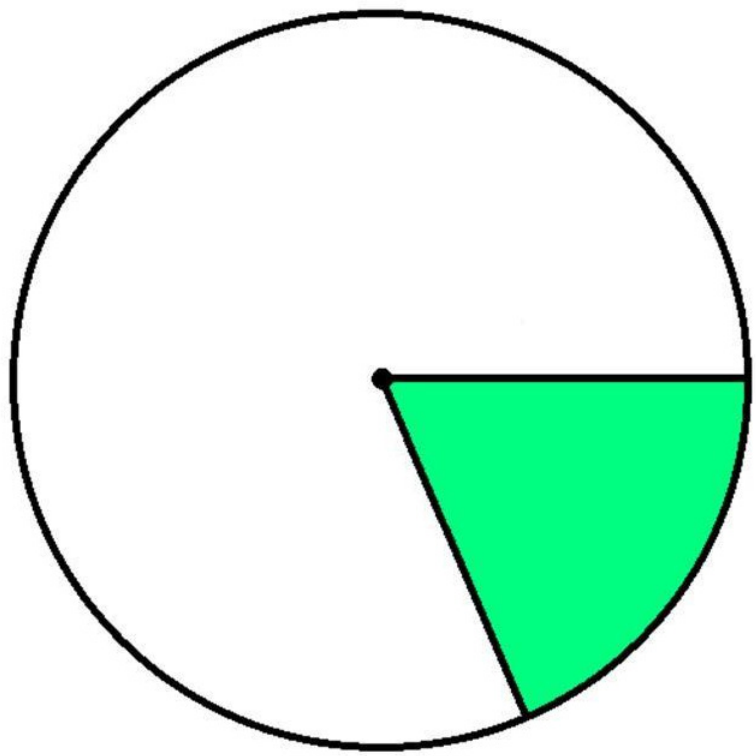
Since the perpendicular from the apex of the cone to the center of the base determines the height h, it follows that we have $q^2 + h^2 = p^2$. It follows that a reasonable model for the cone is given by the points whose **cylindrical coordinates** (r, θ, z) satisfy the conditions

$$0 \;\leq\; r \;\leq\; q, \quad 0 \;\leq\; \theta \;\leq\; 2\pi \; , \quad z \;=\; \frac{h(q-r)}{q}$$

and a continuous mapping φ from the sector defined by $0 \leq r \leq p$ and $0 \leq \theta \leq 2\pi - \alpha$ is given by $\varphi(r, \theta) = h(q - r)/q$.

The gluing instructions imply that the equivalence classes for $\mathcal{R}$ with more than one element are given in polar coordinates by the pairs $\{(r, 0), (r, 2\pi - \alpha)\}$ where $0 < r \leq p$. To complete the verification, one needs to check that φ is onto and if $\varphi(r, \theta) = \varphi(r'\theta')$ then $(r, \theta) \, \mathcal{R} \, (r'\theta')$. This is straightforward but somewhat tedious, and it depends upon the fact that different polar coordinates represent different points of the plane if $r > 0$ and $0 \leq \theta \leq 2\pi - \alpha$. It then follows that φ determines a homeomorphism from the quotient of the circular sector in the plane to the lateral cone surface in $\mathbb{R}^3$.

Constructing a cone from a circular sector



The surface of a cube

Ths solid cube is defined to be $[0,1]^3$, and its boundary surface consists of all $(x,y,z) \in [0,1]^3$ such that at least one of x,y,z is equal to either 0 or 1. One standard way of constructing the boundary surface is to start with the configuration X of squares outlined in the drawing on the next page; formally, X is the union of $[-2,2] \times [0,1]$ and $[0,1] \times [-1,2]$, and two boundary edges are identified if they are drawn in the same color.

We can analyze this example by the same general method as before. The first step is to describe the identifications explicitly, and we shall do so working counterclockwise from $(2,1)$:

The point $(t,1) \in [1,2] \times \{1\}$ is identified with $(1,t) \in \{1\} \times [1,2]$.
The point $(t,2) \in [0,1] \times \{2\}$ is identified with $(-1-t,1) \in [-2,-1] \times \{1\}$.
The point $(1-t,0) \in \{0\} \times [1,2]$ is identified with $(1,t) \in [-1,0] \times \{1\}$.
The point $(-2,t) \in \{-2\} \times [0,1]$ is identified with $(2,t) \in \{2\} \times [0,1]$.
The point $(-1-t,0) \in \{0\} \times [-2,-1] \times \{0\}$ is identified with $(t,-1) \in [0,1] \times \{-1\}$.
The point $(t,0) \in [-1,0] \times \{0\}$ is identified with $(0,t) \in \{0\} \times [-1,0]$.
The point $(1,t) \in \{1\} \times [-1,0]$ is identified with $(1-t,0) \in [0,2] \times \{0\} \times [-1,0]$.

By construction, each point in an open face (a solid square with the boundary removed) determines a one point equivalence class, and each point on an open edge (endpoints removed) lies in a two point equivalence class. Finally, the vertices lie in 8 equivalence classes as follows:

$$\{(0,0)\},\ \{(0,1)\},\ \{(1,0)\},\ \{(1,1)\},\ \{(0,2),(-1,1)\},\ \{(0,-1),(-1,0)\},$$
$$\{(2,1),(1,2),(-2,1)\},\ \{(2,0),(1,-1),(-2,0)\}$$

In order to complete the verification, we need to define a continuous mapping from X to the boundary of the cube which induces a 1–1 correspondence from $X/\mathcal{R}$ to the latter. We shall only give the definition. This will be done individually on each square, so it will be necessary to verify that the definitions agree at the points where two different squares in X intersect (which is enough to say that the function is continuous on X. Then it is necessary to show that this map determines a 1–1 correspondence as above. This verification is entirely elementary, but it is also quite tedious, so we shall not give any details.

So here is the definition of f:

The square $[0,1] \times [0,1]$ maps to the face defined by $z = 0$ via the map sending (u,v) to $(u,v,0)$.

The square $[1,2] \times [0,1]$ maps to the face defined by $x = 1$ via the map sending (u,v) to $(1,v,u-1)$.

The square $[-1,0] \times [0,1]$ maps to the face defined by $x = 0$ via the map sending (u,v) to $(0,v,-u)$.

The square $[-2,-1] \times [0,1]$ maps to the face defined by $z = 1$ via the map sending (u,v) to $(-1-u,v,1)$.

The square $[0,1] \times [-1,0]$ maps to the face defined by $y = 0$ via the map sending (u,v) to $(u,0,-v)$.

The square $[0,1] \times [1,2]$ maps to the face defined by $y = 1$ via the map sending (u,v) to $(u,1,v-1)$.

The reader is strongly urged to cut out a configuration of 6 squares as drawn on the next page, to fold it along the edges of the squares, and to tape the edges together so that the surface of a cube is formed. This might help clarify some of the constructions given above.

Constructing a cube surface from a configuration of squares in the plane