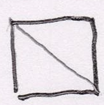
II. Constructing and Deconstructing Spaces

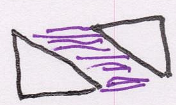
Build an object out of simple pieces, or recognize that an object is built out of simple pieces.

A basic theme in topology and geometry all the way from children's toys to the frontiers of research.

SOLID SQUARE

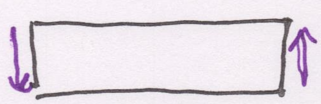


2 SOLID TRIANGLES

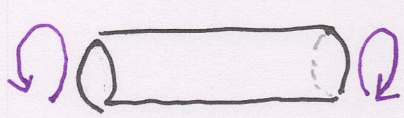


PURPLE = GLUE TOGETHER

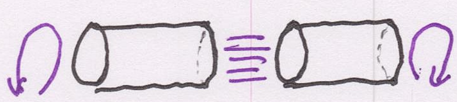
MÖBIUS STRIP



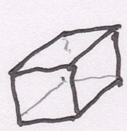
2 SOLID SQUARES



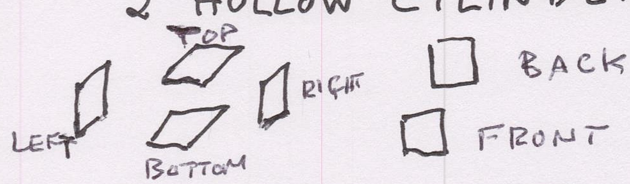
KLEIN BOTTLE



2 HOLLOW CYLINDERS

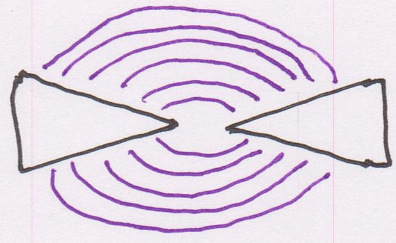


SURFACE OF CUBE



6 SOLID SQUARES GLUED ALONG EDGES

LATERAL SURFACE OF CONE



2 SOLID TRIANGLES

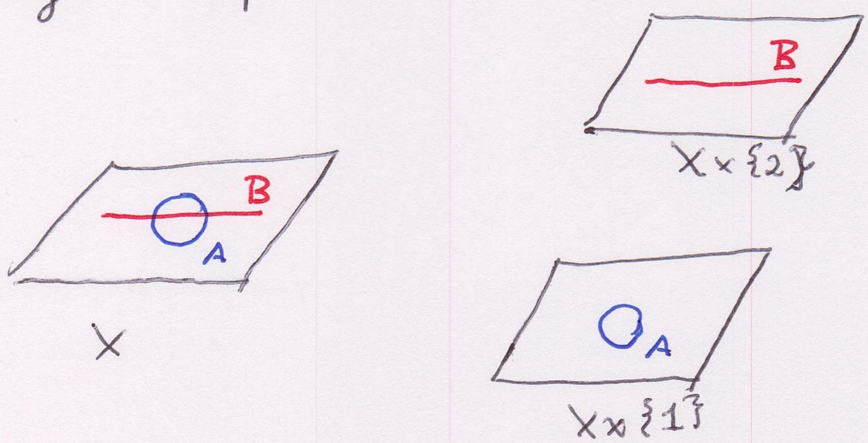
There are two underlying mathematical issues:

① If X is a $\begin{Bmatrix} \text{set} \\ \text{space} \end{Bmatrix}$ and $X = A \cup B$, construct homeomorphic disjoint copies of A and B.

② Given a space X and an equivalence relation $\mathcal{R}$ on X, find the "right" way to put a topological structure on the quotient set $X/\mathcal{R}$.

II. 1 Disjoint unions

$X = A \cup B$ Want to construct disjoint copies of A and B



<u>Definition</u> $A \amalg B = A \times \{1\} \cup B \times \{2\}$

upside down capital Pi

Given topologies $\mathcal{T}_A$, $\mathcal{T}_B$ on A and B, take $\mathcal{T}_A \amalg \mathcal{T}_B$ (disjoint union topology) to be all $C \times \{1\} \cup D \times \{2\}$ where C is open in A and D is open in B. Then the closed subsets have the form $E \times \{1\} \cup F \times \{2\}$ with $\left\{ \begin{matrix} E \text{ closed in } A \\ F \text{ closed in } B \end{matrix} \right\}$.

$i_A : A \longrightarrow A \amalg B \qquad i_A(a) = (a, 1)$

$i_B : B \longrightarrow A \amalg B \qquad i_B(b) = (b, 2)$

$\begin{Bmatrix} i_A \\ i_B \end{Bmatrix}$ maps $\begin{Bmatrix} A \\ B \end{Bmatrix}$ homeomorphically onto its image, if A & B have topological structures and $A \amalg B$ has the disjoint union structure.

Note $i_A[A]$ & $i_B[B]$ are open-and-closed subsets of $A \amalg B$.

Proposition II.1.1 A, B, Y topological spaces.

(i) If $f : A \longrightarrow Y$ and $g : B \longrightarrow Y$ are cont., then there is a unique $h : A \amalg B \longrightarrow Y$ such that $f = h \circ i_A$, $g = h \circ i_B$.

(ii) If $h : A \amalg B \longrightarrow Y$ is a map of sets, then h is continuous $\Longleftrightarrow$ $h \circ i_A$ and $h \circ i_B$ are.

Proof. (i) Take $h(a, 1) = f(a)$ and $h(b, 2) = g(b)$; then $f = h \circ i_A$, $g = h \circ i_B$. Suppose h' also satisfies these

Then $h'(a,1) = h' i_A(a) = f(a)$ and

$$h'(b,2) = h' i_B(b) = g(b),$$ so

$h' = h$ at all points of $A \sqcup B$. ∎

(ii) ($\Longrightarrow$) Composites of continuous maps are continuous. ($\Longleftarrow$) Let $W \subseteq A \sqcup B$ be open. Then $W = U \sqcup V$ where U is open in A and V is open in B. Furthermore, by construction we have $h^{-1}[W] = f^{-1}[U] \sqcup g^{-1}[V]$ (check this!). Since f and g are continuous we know that $f^{-1}[U]$ is open in A and $g^{-1}[V]$ is open in B, so the disjoint union is open in $A \sqcup B$. Thus h satisfies the definition of continuity for topological spaces. ∎

<u>Proposition II. 1. 2</u> (Internal disjoint unions)

Let X be a topological space such that $X = A \cup B$, where $A \cap B = \emptyset$ and both are $\begin{bmatrix} \text{closed} \\ \text{open} \end{bmatrix}$ in X. Let $j_A, j_B : A, B \longrightarrow X$ denote the inclusions.

Then there is a unique homeomorphism $\varphi : A \sqcup B \longrightarrow X$ such that $\varphi \circ i_A = j_A$ and $\varphi \circ i_B = j_B$.

Like the difference between vector spaces being internal direct sums of two subspaces vs. being an external direct sum of two spaces.

<u>Example</u> $\mathbb{R} - \{0\}$ is an internal disjoint union of $(-\infty, 0)$ and $(0, \infty)$. <u>But</u> $[0,1]$ is not homeomorphic to $[0, \frac{1}{2}) \sqcup [\frac{1}{2}, 1]$!

<u>PROOF OF II.1.2</u> The preceding proposition yields a unique continuous map with the given composition properties. We need to show φ is a homeomorphism. We know <u>φ is continuous.</u>

<u>φ is 1-1</u> The image of φ is $A \cup B$, so if there is some $y \in X$ with $\varphi(u) = \varphi(v) = y$ then either $y \in A$ or $y \in B$ but not both. This means $\{u, v\} \subseteq A \times \{1\}$ or $\{u, v\} \subseteq B \times \{2\}$. By construction φ is 1-1 on both $A \times \{1\}$ and $B \times \{2\}$, so we must have $u = v$.

<u>φ is onto</u> We have $A, B \subseteq \text{Image}\,\varphi$ so $X \supseteq \text{Image}\,\varphi \supseteq A \cup B = X$.

<u>φ is open</u> Let $W \subseteq A \amalg B$ be open. Then $W = U \amalg V$, where U is open in A and V is open in B. Furthermore, $\varphi[W] = U \cup V$. Now U open in A & A open in $X \Rightarrow U$ open in X and V open in B & B open in $X \Rightarrow V$ open in X, so $U \cup V = \varphi[W]$ is open in X, as required. ■

<u>Disjoint $\left\{\begin{matrix}\text{unions}\\ \text{sums}\end{matrix}\right\}$ and topological properties</u>

A, B compact $\Rightarrow A \amalg B$ compact
A, B Hausdorff $\Rightarrow A \amalg B$ Hausdorff (and conversely)

$A \amalg B$ is connected $\Leftrightarrow$ one is empty and the other is connected.

(Crossley, pp. 70-71).

<u>Note</u> The latter shows that $[0, \frac{1}{2}) \amalg [\frac{1}{2}, 1]$ is not connected and hence not homeomorphic to $[0, 1]$.