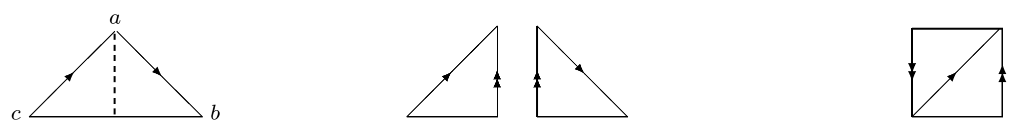
Solutions to Chapter 15 exercises

15.1 (a) A figure-of-eight.

(b) A (two-dimensional) sphere. (See below.)

(c) This gives a Möbius band. (See below.) For we first cut along the dashed vertical line in the triangle abc to get two triangles as in the middle picture, with vertical sides labelled to recall how these are to be stuck together. Then we reassemble these middle two triangles to get the picture on the right, which represents a Möbius band.



15.2 Recall that (ii) and (iii) together are equivalent to (iv): $\{s_1, s_2\} = \{0, 1\}$ and $t_2 = 1 - t_1$.

Reflexivity We have to show that $(s, t) \sim (s, t)$ for any $(s, t) \in [0, 1] \times [0, 1]$. But this follows from (i).

Transitivity Suppose that $(s_1, t_1) \sim (s_2, t_2)$ and $(s_2, t_2) \sim (s_3, t_3)$. We want to prove that $(s_1, t_1) \sim (s_3, t_3)$.

If (i) holds for (s_1, t_1) and (s_2, t_2), so $s_1 = s_2$ and $t_1 = t_2$, then $(s_1, t_1) = (s_2, t_2) \sim (s_3, t_3)$, so $(s_1, t_1) \sim (s_3, t_3)$. Similarly if $s_2 = s_3$ and $t_2 = t_3$ then $(s_1, t_1) \sim (s_3, t_3)$.

If (iv) holds for both pairs (s_1, t_1) (s_2, t_2) and (s_2, t_2) (s_3, t_3), then we have

$$\{s_1, s_2\} = \{0, 1\} \text{ and } t_2 = 1 - t_1, \quad \{s_2, s_3\} = \{0, 1\} \text{ and } t_3 = 1 - t_2.$$

These give either $s_1 = 0$ and $s_2 = 1$ so $s_3 = 0 = s_1$, or $s_1 = 1$ and $s_2 = 0$ so $s_3 = 1 = s_1$, and also $t_3 = 1 - t_2 = 1 - (1 - t_1) = t_1$. Hence $s_1 = s_3$ and $t_1 = t_3$, so $(s_1, t_1) \sim (s_3, t_3)$ by (i).

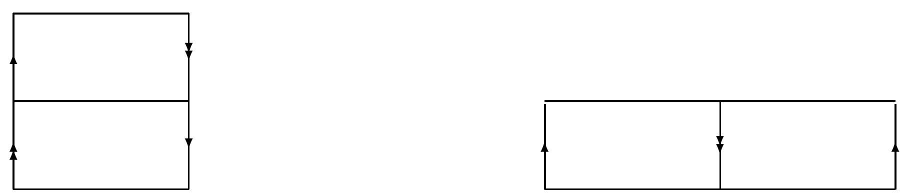
15.3 First, it is clear that $(0, t) \sim (1, 1 - t)$ for any $t \in [0, 1]$, by (iv). What needs proving is that the equivalence classes are no bigger than the sets described in the question.

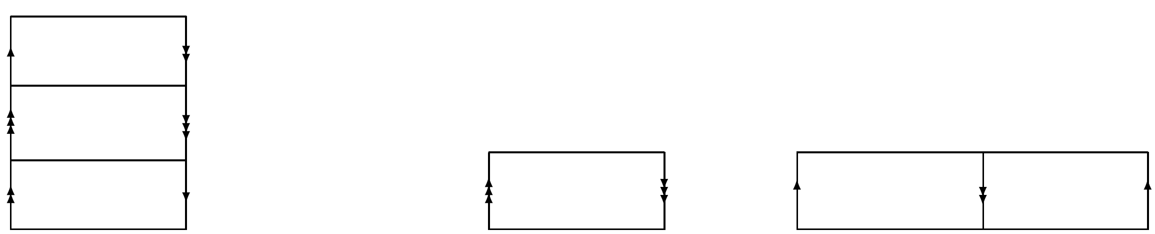
First if $0 < s < 1$ and $(s_2, t_2) \sim (s, t)$ then (i) must apply and $(s_2, t_2) = (s, t)$. So (s, t) cannot be equivalent to any point other than itself, and $\{(s, t)\}$ is a complete equivalence class.

Consider now which points can be equivalent to $(0,\, t)$, for some $t \in [0,\, 1]$. So suppose that $(s_2,\, t_2) \sim (0,\, t)$. Then *either* (i) applies and $(s_2,\, t_2) = (0,\, t)$, *or* (ii) applies and $s_2 = 1$ and $t_2 = 1 - t$. Thus the set $\{(0,\, t),\, (1,\, 1-t)\}$ is a complete equivalence class.

Thus each equivalence class is either a singleton set $\{(s,\, t)\}$ with $0 < s < 1$ and $t \in [0,\, 1]$, or a set containing the two elements $\{(0,\, t),\, (1,\, 1-t)\}$ for some $t \in [0,\, 1]$.

15.4 We answer this by showing in the diagrams below what happens. There is an underlying assumption: that 'cutting along a line' removes that line. In each diagram, the left-hand picture shows what we begin with, marking the lines to be cut along, and the right-hand picture shows what happens after the cutting.





These show that the required outcomes occur.

15.5 For example since $[0,\, \pi) = (-\infty, \pi) \cap [0,\, 2\pi]$ it follows that $[0,\, \pi)$ is open in $[0,\, 2\pi]$. Now $f([0,\, \pi))$ is the upper semi-circle of S^1, closed at the end with coordinates $(1,\, 0)$ and open at the end with coordinates $(-1,\, 0)$. This is not open in S^1, since any open set in S^1 containing the point $(1,\, 0)$ also contains points $(x,\, y) \in S^1$ with $y < 0$, and such points are not in $f([0,\, \pi))$.

15.6 This is an equivalence relation by Exercise 2.7. There are just two equivalence classes, the set of rationals $\mathbb{Q}' = \mathbb{Q} \cap [0,\, 1]$ in $[0,\, 1]$ and the set of irrationals $\mathbb{I}' = [0,\, 1] \setminus \mathbb{Q}' = (\mathbb{R} \setminus \mathbb{Q}) \cap [0,\, 1]$ in $[0,\, 1]$. So the corresponding quotient space $[0,\, 1]/\!\sim$ has exactly two points. To show that this quotient space has the indiscrete topology we need to show that the singleton subsets of $[0,\, 1]/\!\sim$ are not open. By definition of the quotient topology this amounts to seeing that $\mathbb{Q}'$ and $\mathbb{I}'$ are not open in $[0,\, 1]$. But this is a familiar fact.

15.7 This is a matter of taking complements. By Proposition 3.9 $X \setminus f^{-1}(V) = f^{-1}(Y \setminus V)$ for any subset $V \subseteq Y$. Suppose that $f : X \to Y$ is a quotient map. Then $V \subseteq Y$ is closed in Y iff $Y \setminus V$ is open in Y iff $X \setminus f^{-1}(V) = f^{-1}(Y \setminus V)$ is open in X iff $f^{-1}(V)$ is closed in X. Conversely suppose that $V \subseteq Y$ is closed in Y iff $f^{-1}(V)$ is closed in X. Then $U \subseteq Y$ is open in Y iff $Y \setminus U$ is closed in Y iff $X \setminus f^{-1}(U) = f^{-1}(Y \setminus U)$ is closed in X iff $f^{-1}(U)$ is open in X, so f is a quotient map.

15.8 Suppose that $f : X \to Y$ and $g : Y \to Z$ are quotient maps. Recall this means, for example for f, that f is onto and $U \subseteq Y$ is open in Y iff $f^{-1}(U)$ is open in X.

First, $g \circ f : X \to Z$ is onto since both f and g are onto. Now consider any subset $U \subseteq Z$. Since g is a quotient map, U is open in Z iff $g^{-1}(U)$ is open in Y. Since f is also a quotient map, $g^{-1}(U)$ is open in Y iff $f^{-1}(g^{-1}(U))$ is open in X, which happens iff $(g \circ f)^{-1}(U)$ is open in X. Hence U is open in Z iff $(g \circ f)^{-1}(U)$ is open in X. We have now proved that $g \circ f$ is a quotient map.

15.9 We proceed as in the proof that the quotient space of the square $S = [0, 2\pi] \times [0, 2\pi]$ by an appropriate equivalence relation is homeomorphic to T. Define
$f : \mathbb{R}^2 \to T$ by $f(s, t) = ((a + r\cos t)\cos s, (a + r\cos t)\sin s, r\sin t)$. Then as in the proof mentioned above, we show that
(a) f is a map to T,
(b) $i \circ f : \mathbb{R}^2 \to \mathbb{R}^3$ is continuous, where $i : T \to \mathbb{R}^3$ is the inclusion, so f is continuous by Proposition 10.6.
(c) f respects the equivalence relation $\sim$ just as before.

It follows that f induces a well-defined continuous map $g : \mathbb{R}^2/{\sim} \to T$. Just as before (in the proof with S replacing $\mathbb{R}^2$), we can show that g is one-one onto T. Since T is Hausdorff as a subspace of $\mathbb{R}^3$, it remains to prove that $\mathbb{R}^2/{\sim}$ is compact, and this is really the only difference from the proof when $\mathbb{R}^2$ is replaced by S. But if j denotes the inclusion $j : S \to \mathbb{R}^2$, then the composition $p \circ j$ is onto, where $p : \mathbb{R}^2 \to \mathbb{R}^2/{\sim}$ is the quotient map; for given any point $(s, t) \in \mathbb{R}^2$ we can find a point $(s', t') \in S$ such that $p((s', t')) = p((s, t))$. [If S were the square $[0, 1] \times [0, 1]$ we could take $s' = s - [s]$ and $t' = t - [t]$ where $[s]$, $[t]$ are the integer parts of s, t. Since S is actually $[0, 2\pi] \times [0, 2\pi]$ we scale this and take $s' = 2\pi(s/2\pi - [s/2\pi])$ and $t' = 2\pi(t/2\pi - [t/2\pi])$. Hence since S is compact, so is $\mathbb{R}^2/{\sim}$, and as before we can apply Corollary 13.27 to see that g is a homeomorphism.