Comments on the Brouwer Fixed Point Theorem

We shall give a complete and rigorous proof of a key step in the proof of the Brouwer Fixed Point Theorem that is often treated only in an intuitive manner. Everything will be stated in terms of the 2-dimensional disk D^2 and its boundary S^1, but the entire argument extends to cover disks and spheres of higher dimensions.

The retraction in Brouwer's Theorem

The idea which appears in most books is straightforward. If we have a continuous mapping $f : D^2 \to D^2$ with no fixed points, then $f(\mathbf{x}) \neq \mathbf{x}$ for all $\mathbf{x}$, and hence it is meaningful to discuss the ray starting with $f(\mathbf{x})$ which passes through $\mathbf{x}$. Simple pictures strongly suggest that there is a unique point $r(\mathbf{x})$ on this ray which lies on the boundary circle and that this point depends continuously on $\mathbf{x}$. If $\mathbf{x}$ already lies on the circle, then this point is $\mathbf{x}$ itself, so we have a continuous mapping $r : D^2 \to S^1$ such that $r|S^1$ is the identity. One then derives a contradiction using the fact that S^1 and D^2 are not homotopy equivalent.

Of course, it is absolutely necessary to prove that one actually obtains a continuous mapping r with the indicated properties. We shall take a more general approach, starting with two distinct points $\mathbf{x}$ and $\mathbf{y}$ on the disk D^n and considering the ray starting with $\mathbf{y}$ and passing through $\mathbf{x}$; algebraically, this is the set of all points expressible as $\mathbf{y} + (1-t)\mathbf{x}$, where $t \geq 0$. As before, simple pictures strongly suggest that

(1) there is a unique scalar $t \geq 1$ such that $\mathbf{y} + (1-t)\mathbf{x}$ lies on S^{n-1},

(2) if $\mathbf{x} \in S^{n-1}$ then $t = 1$ so that the point is equal to $\mathbf{x}$, and

(3) the value of t is a continuous function of $(\mathbf{x}, \mathbf{y})$.

Our purpose here is to justify these assertions.

PROPOSITION. *There is a continuous function* $\rho : D^2 \times D^2 - \text{Diagonal} \to S^1$ *such that* $\rho(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ *if* $\mathbf{x} \in S^1$.

If we have the mapping ρ and f is a continuous map from D^2 to itself without fixed points, then the retraction from D^2 onto S^1 is given by $\rho(\mathbf{x}, f(\mathbf{x})\,)$.

Proof of the proposition. It follows immediately that the intersection points of the line joining $\mathbf{y}$ to $\mathbf{x}$ are give by the values of t which are roots of the equation

$$|\mathbf{y} \; + \; t(\mathbf{x} - \mathbf{y})|^2 \quad = \quad 1$$

and the desired points on the ray are given by the roots for which $t > 1$. We need to show that there is always a unique root satisfying this condition, and that this root depends continuously on $\mathbf{x}$ andn $\mathbf{y}$.

We can rewrite the displayed equation as

$$|\mathbf{x} - \mathbf{y}|^2 t^2 \; + \; 2\langle \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle t \; + \; (|\mathbf{y}|^2 - 1) \quad = \quad 0 \; .$$

If try to solve this nontrivial quadratic equation for t using the quadratic formula, then we obtain the following:

$$t \quad = \quad \frac{-\langle \mathbf{y}, \mathbf{x}-\mathbf{y}\rangle \ \pm \ \sqrt{\langle \mathbf{y}, \mathbf{x}-\mathbf{y}\rangle^2 \ + \ |\mathbf{x}-\mathbf{y}|^2\cdot(1-|\mathbf{y}|^2)}}{|\mathbf{x}-\mathbf{y}|^2}$$

One could try to analyze these roots by brute force, but it will be more pleasant to take a more qualitative viewpoint.

(*a*) *There are always two distinct real roots.* We need to show that the expression inside the square root sign is always a positive real number. Since $|\mathbf{y}| \leq 1$, the expression is clearly nonnegative, so we need only eliminate the possibility that it might be zero. If this happens, then each summand must be zero, and since $|\mathbf{y}-\mathbf{x}| > 0$ it follows that we must have both $\langle \mathbf{y}, \mathbf{x}-\mathbf{y}\rangle = 0$ and $1 - |\mathbf{y}|^2 = 0$. The second of these implies $|\mathbf{y}| = 1$, and the first then implies

$$\langle \mathbf{y}, \mathbf{x}\rangle \quad = \quad |\mathbf{y}|^2 \quad = \quad 1\ .$$

If we combine this with the Cauchy-Schwarz Inequality and the basic condition $|\mathbf{x}| \leq 1$, we see that $|\mathbf{x}|$ must equal 1 and $\mathbf{x}$ must be a positive multiple of $\mathbf{y}$; these in turn imply that $\mathbf{x} = \mathbf{y}$, which contradicts our hypothesis that $\mathbf{x} \neq \mathbf{y}$. Thus the expression inside the radical sign is positive and hence there are two distinct real roots.

(*b*) *There are no roots t such that $0 < t < 1$.* The Triangle Inequality implies that

$$|\mathbf{y} \ + \ t(\mathbf{x}-\mathbf{y})| \quad = \quad |(1-t)\mathbf{y} \ + \ t\mathbf{x}| \quad \leq \quad (1-t)|\mathbf{y}| \ + \ t|\mathbf{x}| \quad \leq 1$$

so the value of the quadratic function

$$q(t) \quad = \quad |\mathbf{x}-\mathbf{y}|^2t^2 \ + \ 2\langle \mathbf{y}, \mathbf{x}-\mathbf{y}\rangle t \ + \ (|\mathbf{y}|^2 - 1)$$

lies in $[-1, 0]$ if $0 < t < 1$. Suppose that the value is zero for some t_0 of this type. Since there are two distinct roots for the associated quadratic polynomial, it follows that the latter does not take a maximum value at t_0, and hence there is some t_1 such that $0 < t_1 < 1$ and the value of the function at t_1 is positive. This contradicts our observation about the behavior of the function, and therefore our hypothesis about the existence of a root like t_0 must be false.

(*d*) *There is one root of $q(t)$ such that $t \leq 0$ and a second root such that $t \geq 1$.* We know that $q(0) \leq 0$ and that the limit of $q(t)$ as $t \to -\infty$ is equal to $+\infty$. By continuity there must be some $t_1 \leq 0$ such that $q(t_1) = 0$. Similarly, we know that $q(1) \leq 0$ and that the limit of $q(t)$ as $t \to +\infty$ is equal to $+\infty$, so again by continuity there must be some $t_2 \geq 1$ such that $q(t_2) = 0$.

(*d*) *The unique root t satisfying $t \geq 1$ is a continuous function of $\mathbf{x}$ and $\mathbf{y}$.* This is true because the desired root is given by taking the positive sign in the expression obtained from the quadratic formula, and it is a routine algebraic exercise to check that this expression is a continuous function of $(\mathbf{x}, \mathbf{y})$.

(*e*) *If $|\mathbf{x}| = 1$, then $t = 1$.* This just follows because $|\mathbf{y} \ + \ 1(\mathbf{x}-\mathbf{y})| = 1$ in this case.

The proposition now follows by taking

$$\rho(\mathbf{x}, \mathbf{y}) \quad = \quad \mathbf{y} \ + \ t(\mathbf{x}-\mathbf{y})$$

where t is given as above by taking the positive sign in the quadratic formula. The final property shows that $\rho(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ if $|\mathbf{x}| = 1$.■