# Problems from Munkres for Mathematics 145B—I

Recommended exercises from Sutherland are also listed.

### S.14. Sequentially compact metric spaces

Sutherland, 14.15, 14.17($c$).

## I. Complete metric spaces

Sutherland, 17.2–3, 17.8, 17.10, 17.16.

(Munkres, § 43, pp. 270–271: 1, 4, 6$c$)

**1.** Let $X$ be a metric space.

($a$) Suppose that for some $\varepsilon > 0$, every $\varepsilon$-ball in X has compact closure. Show that $X$ is complete.

($b$) Suppose that for each $x \in X$ there is some $\varepsilon > 0$ such that the disk $N_\varepsilon(x)$ has compact closure. Show by means of an example that $X$ need not be complete. [*Hint:* What happens if we take a nonempty proper subset of $\mathbb{R}^2$?]

**4.** Show that the metric space $(X, d)$ is complete if and only if for every (nonempty) nested sequence $A_1 \supset A_2 \supset \cdots$ of nonempty closed sets of $X$ such that diam $A_n \to 0$, then the intersection of the sets $A_n$ is nonempty.

**6.** A space $X$ is said to be *topologically complete* if there exists a metric for the topology of $X$ relative to which $X$ is complete.

($c$) Show that an open subspace of a topologically complete space is topologically complete. [*Hint:* If $U \subset X$ and $X$ is complete under the metric $d$, define $\phi : U \to \mathbb{R}$ by the formula $1/d(x, XU)$ and embed $U$ in $X \times \mathbb{R}$ by taking the graph of $\phi$.]

## II. Constructing and deconstructing spaces

Sutherland, 15.4 — only the first part.

If you have not seen a physical demonstration of the result in this exercise, it is worthwhile carrying out the construction using a strip of paper, a piece of adhesive tape, and a scissors. After making a Möbius strip by taping the ends together in the proper way, mark off the circle in the middle of the strip, cut along this line, and see what happens.

(Munkres, § 22, pp. 144–145)

**4.** ($a$) Define an equivalence relation on the plane $X = \mathbb{R}^2$ as follows:

$$(x_0, y_0) \quad \sim \quad (x_1, y_1) \qquad \Leftrightarrow \qquad x_0 + y_0^2 \quad = \quad x_1 + y_1^2$$

Let $X^*$ be the corresponding quotient space. It is homeomorphic to a familiar space; what is it? [*Hint:* Set $g(x, y) = x + y^2$.]

($b$) Repeat ($a$) for the equivalence relation

$$(x_0, y_0) \quad \sim \quad (x_1, y_1) \qquad \Leftrightarrow \qquad x_0^2 + y_0^2 \quad = \quad x_1^2 + y_1^2 \ .$$

# III. Homotopy

## III.1 : Homotopic mappings

Munkres, § 51, p. 330

**1.** Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h' : X \to Z$ are homotopic.

**2.** Given spaces $X$ and $Y$, let $[X, Y]$ denote the set of homotopy classes of maps from $X$ into $Y$.

($a$) Let $I = [0, 1]$. Show that for any $X$, the set $[X, I]$ has a single element.

($b$) Show that if $Y$ is path connected, the set $[I, Y]$ has a single element.

**3.** A space $X$ is said to be *contractible* if the identity map $\mathrm{id}_X : X \to X$ is nullhomotopic.

($a$) Show that $I$ and $\mathbb{R}$ are contractible.

($b$) Show that a contractible space is path connected.

($c$) Show that if $Y$ is contractible, then for all $X$, the set $[X, Y]$ has a single element.

($d$) Show that if $X$ is contractible and $Y$ is path connected, then $[X, Y]$ has a single element.

## III.2 : Homotopy equivalence

Munkres, § 58, pp. 366–367

**1.** Show that if $A$ is a deformation retract of $X$, and $B$ is a deformation retract of $A$, then $B$ is a deformation retract of $X$.

**3.** Show that, given a collection $\mathcal{C}$ of spaces, the relation of homotopy equivalence is an equivalence relation on $\mathcal{C}$.

**6.** Show that a retract of a contractible space is contractible.