III. Homotopy

Intuitively, a continuous deformation over a period of time.

(See the <u>Wikipedia</u> <u>gif</u> for the topic.)

Formally,

<u>Def.</u> f, g continuous functions $X \to Y$. Then f and g are <u>homotopic</u>* ($f \simeq g$) if there is some $H : X \times [0,1] \longrightarrow Y$ continuous such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. The map H is called a <u>homotopy</u>* from f to g, and sometime we shall write $H : f \simeq g$ or $f \underset{H}{\simeq} g$.

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* <u>Pronunciation</u>: HOME-oh-<u>top</u>-ee

<u>home</u>-oh-TOP-ic

underline means secondary emphasis.

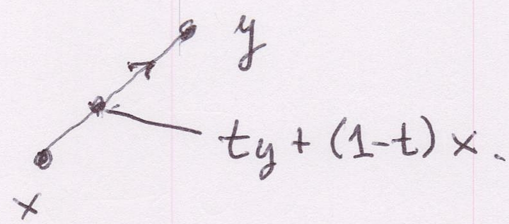
Fact 1 Let X be a compact space, and let $U \subseteq \mathbb{R}^n$ be open. Given a cont. function $f: X \to U$ there is some $\delta > 0$ such that $\|f - g\| < \delta \Rightarrow f$ is homotopic to g.

Fact 2 If $X \subseteq \mathbb{R}^m$ for some m, then there is a countable sequence of maps $\{h_k\}$ such that every $f: X \to U$ is homotopic to one of these mappings h_k.

III.1 Basic Concepts

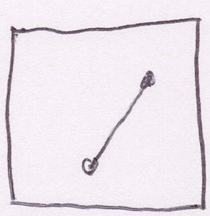
<u>Background</u> In $\mathbb{R}^n$, if $x \neq y$ then the closed line segment joining x to y is the curve

$$\gamma(t) = (1-t)x + ty \qquad 0 \le t \le 1$$



<u>Def.</u> A subset $Y \subseteq \mathbb{R}^n$ is <u>convex</u> if for each $y_1, y_2 \in Y$ and t such that $0 \le t \le 1$ we have $(1-t)y_1 + t y_2 \in Y$.

"No dents or holes"



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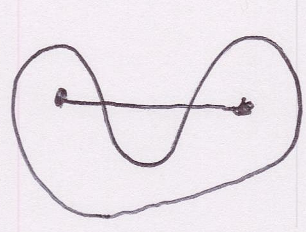

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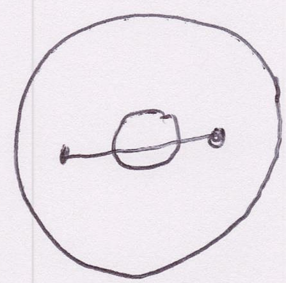

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<u>Proposition III. 1. 1</u> If Y_1 and Y_2 are convex, then so is $Y_1 \cap Y_2$. ∎

<u>Proposition III. 1. 2</u> If Y is convex and $f, g : X \longrightarrow Y$ are continuous, then $f \simeq g$.

<u>Proof of III. 1. 2</u> Let $H(x,t) = t\,g(x) + (1-t) f(x)$. Then $H : f \simeq g$. ∎

<u>EXAMPLE</u> $X = Y = \{0,1\}$, $f =$ identity, $g(x) = 0$ all x. Then f and g are not homotopic.

<u>Verification</u> Suppose $H : f \simeq g$. Then $\gamma(t) = H(1,t)$ defines a continuous curve from ~~[illegible]~~ $1 = \gamma(0)$ to $0 = \gamma(1)$.

Therefore 1 and 0 lie in a connected subset of $\{0,1\}$. But this is false; the source of the contradiction is the assumption that $f \simeq g$, so this must be false. ■

<u>NOTE</u> Two continuous maps $f \,\&\, g : \{pt.\} \longrightarrow Y$ (any Y) are homotopic $\Longleftrightarrow$ the points $f(pt.)$ and $g(pt.)$ lie in the same $\left\{\begin{matrix} \text{arc} \\ \text{path} \end{matrix}\right\}$ component of Y.

<u>Homotopy is an equivalence relation</u>

<u>$f \simeq f$</u> Let $H : X \times [0,1] \longrightarrow Y$ be $H(x,t) = f(x)$

<u>$f \simeq g \Rightarrow g \simeq f$</u> Let $H : f \simeq g$, and define $K(x,t) = H(x, 1-t)$. (going backwards in time).

Then $K : g \simeq f$.

$f \simeq g$ and $g \simeq h \Rightarrow f \simeq h$ Say $K: f \simeq g$ and $L: g \simeq h$. Concatenate (string together) the homotopies: Use K to define a homotopy for $0 \leq t \leq \frac{1}{2}$, use L to define one for $\frac{1}{2} \leq t \leq 1$. Formally,

$$\Gamma(x,t) = \begin{cases} K(x, 2t) & 0 \leq t \leq 1/2 \\ L(x, 2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

Note that $K(x,1) = L(x,0)$ by hypothesis.

RECALL Suppose $X = A \cup B$ with A, B $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ in X, and let $\begin{matrix} f: A \to Y \\ g: B \to Y \end{matrix}$ be cont. If $f|A \cap B = g|A \cap B$, then there is a unique continuous function $h: X \to Y$ such that $h|A = f$ and $h|B = g$.

By Fact 2 above, this says that the continuous family of continuous functions is simplified to a countable object if we take homotopy classes of mappings $X \to U$, where $\left\{\begin{matrix} X \text{ is compact in } \mathbb{R}^m \\ U \text{ is open in } \mathbb{R}^n \end{matrix}\right\}$.

Homotopy and compositions

① $f_0 \simeq f_1 : X \to Y$, $g : Y \to Z$ cont $\Rightarrow g \circ f_0 \simeq g \circ f_1$.

② $f : X \to Y$, $g_0 \simeq g_1 : Y \to Z$ cont. $\Rightarrow g_0 \circ f \simeq g_1 \circ f$.

③ COR. $f_0 \simeq f_1$ + $g_0 \simeq g_1 \Rightarrow g_0 f_0 \simeq g_1 f_1$.

Derivations of ① and ②

① If $H : f_0 \simeq f_1$, then $g \circ H : g \circ f_0 \simeq g \circ f_1$. ∎

② If $K : g_0 \simeq g_1$, then $K \circ f \times id_{[0,1]} : g_0 f \simeq g_1 f$. ∎

By ③ it is meaningful to talk about composing homotopy classes.