Addendum – Arc Components

X space. Write $x_0 \sim x_1 \Longleftrightarrow$ there is a continuous curve $\gamma : [0,1] \longrightarrow X$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$.

<u>CLAIM</u> $\sim$ is an equivalence relation.

$x \sim x$ by the constant curve; $x \underset{\gamma}{\sim} y \Rightarrow y \underset{\beta}{\sim} x$ where $\beta(t) = \gamma(1-t)$; $x \underset{\alpha}{\sim} y$ and $y \underset{\beta}{\sim} z \Rightarrow x \underset{\alpha+\beta}{\sim} z$ where $\alpha+\beta$ is the concatenation $\alpha+\beta(t) = \begin{cases} \alpha(2t) & t \le \frac{1}{2} \\ \beta(2t-1) & t \ge \frac{1}{2} \end{cases}$. ∎

The equivalence classes are called <u>arc components.</u>

<u>CLAIM</u> Arc components are maximal arcwise connected subsets.

An arc component is arcwise connected because $x \sim y \Rightarrow$ there is a continuous curve γ joining x to y, and $\gamma(t) \sim x$ for all t in $[0,1]$.

If C is an arc component and $C \subseteq D$ where D is arcwise connected, then $y \in D \Rightarrow$ y can be joined to a point of C by a curve, which means that y lies in the equivalence class C. ■

<u>Prop.</u> $f: X \to Y$ continuous, X arcwise connected $\Rightarrow$ so is $f[X]$.

<u>Proof</u> Let $y_0, y_1 \in f[X]$ and write $y_i = f(x_i)$. Let γ be a curve in X joining x_0 to x_1. Then $f \circ \gamma$ is a curve in Y joining y_0 to y_1. ■